DIRECTORATE OF DISTANCE EDUCATION

UNIVERSITY OF NORTH BENGAL

MASTER OF SCIENCES- MATHEMATICS

SEMESTER -IV

ALGEBRAIC TOPOLOGY

DEMATH4ELEC6

BLOCK-2

UNIVERSITY OF NORTH BENGAL

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FOREWORD

The Self Learning Material (SLM) is written with the aim of providing simple and organized study content to all the learners. The SLMs are prepared on the framework of being mutually cohesive, internally consistent and structured as per the university's syllabi. It is a humble attempt to give glimpses of the various approaches and dimensions to the topic of study and to kindle the learner's interest to the subject

We have tried to put together information from various sources into this book that has been written in an engaging style with interesting and relevant examples. It introduces you to the insights of subject concepts and theories and presents them in a way that is easy to understand and comprehend.

We always believe in continuous improvement and would periodically update the content in the very interest of the learners. It may be added that despite enormous efforts and coordination, there is every possibility for some omission or inadequacy in few areas or topics, which would definitely be rectified in future.

We hope you enjoy learning from this book and the experience truly enrich your learning and help you to advance in your career and future endeavours.

ALGEBRAIC TOPOLOGY

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Unit-2 The fundamental group

Unit-3 Properties of fundamental group

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BLOCK-2 ALGEBRAIC TOPOLOGY

Algebraic Topology is an important branch of topology having several connections with many areas of modern mathematics. Its growth and influence, particularly since the early forties of the twentieth century, has been remarkably high.

It is best suited for those who have already had an introductory course in topology as well as in algebra. Experience suggests that a comprehensive coverage of the topology of simplicial complexes, simplicial homology of polyhedra, fundamental groups, covering spaces and some of their classical applications like invariance of dimension of Euclidean spaces, Brouwer's Fixed Point Theorem, etc. are the essential minimum which must find a place in a beginning course on algebraic topology. Having learnt these basic concepts and their powerful techniques, one can then go on in any direction of the subject at an advanced level depending on one's interest and requirement.

In block 2 we explain the topology of simplicial complexes, introduce thenotion of barycentric subdivision and then prove the simplicial approximation theorem. We introduce the first classical homology theory, viz., the simplicial homology of a simplicial complex and then proceed to define the simplicial homology of a compact polyhedron. We explain about applications of the homology spectral sequence and algebraic curves.

UNIT-8 GENERAL BORDISM THEORIES

STRUCTURE

8.0 Objective

8.1 Introduction

8.2 General Bordism theories

8.3 Classifying spaces

8.4 Construction of the Thom Spectra

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8.0 OBJECTIVE

In this unit we will learn and understand about general bordism theories, Classifying spaces, Construction of the Thom spectra, Generalized homology theories.

8.1 INTRODUCTION

In this chapter (in contrast to the rest of this book), the word "manifold" will mean a compact, smooth manifold with or without boundary and a submanifold $V \subset M$ will mean a compact submanifold whose boundary is contained in the boundary of M in such a way that V meets the boundary of M transversely. The normal bundle of a submanifold $i: V \to M$ is the quotient bundle $i^*(TM)/TV$ and we will use the notation $v(V \to M)$ or $v(i)$.If M is a submanifold of Rn, or more generally if M has a Riemann-ianmetric, then the normal bundle $v(V \to M)$ can be identified with the sub bundle of $TM|_V$ consisting of all tangent vectors in T_pM which are perpendicular to T_pV , where $P \in V$. A tubular neighborhood of a submanifold $i: V \to M$ is a embedding $f: v(i) \to M$ which M which restricts to the identity on (the zero section) V . Informally, we say that the open set $U = f(v(i)) \subset M$ is a tubular neighborhood of V.

8.2 GENERAL BORDISM THEORIES

Framed bordism is a special case of a general bordism theory, where one consider bordisms respecting some specific stable structure on the normal bundle of a smooth manifold. We will give examples of stable structures now, and then ask you to supply a general definition in Exercise 1.

Basically a property of vector bundles is stable if whenever a bundle η has that property, then so does $\eta \oplus \Sigma^k$ for all k.

8.2.1. Framing. A stable framing on a bundle $[\eta]$ is, as we have seen, a choice of homotopy class of bundle isomorphic

$$\gamma : \eta \oplus \Sigma^k \cong \Sigma^{n+k}$$

Subject to the equivalence relation generated by the requirement that $\gamma \sim \gamma \oplus Id : \eta \oplus \Sigma^k \oplus \Sigma^{n+k+1}$.

8.2.2. The empty structure. This refers to bundles with no extra structure.

8.2.3. Orientation. This is weaker than requiring a framing. The most succient way to define an orientation of a n-plane bundle η of is to choose a homotopy class of trivialization of the highest exterior power of the bundle,

$$\gamma : \wedge^n (\eta) \cong \Sigma.$$

Equivalently, an orientation is a reduction of the structure group to $GL_+(n,R),$ the group of n-by-n matrices with positive determinanat. A oriented manifold is a manifold with an orientation on its tangent bundle.

Since $\wedge^{a+b}(V \oplus W)$ is canonically isomorphic to $\wedge^a V \otimes \wedge^b W$ if V is a a-dimensional vector space and W is a b-dimensional vector space, it follows that $\wedge^n(\eta)$ is canonically isomorphic to $\wedge^{n+k}(\eta \oplus \Sigma^k)$ for any $k \geq 0.$ Thus an orientation on η induces one on $\eta \oplus \Sigma,$ so an orientation is a well-defined stable property.

8.2.4.Spin structure. Let $Spin(n) \to SO(n)$ be the double cover where $Spin(n)$ is connected for n > 1. A spin structure on an n-plane bundle η over a space M is a reduction of the structure group to $Spin(n)$. This is equivalent to giving a principle $Spin(n)$-bundle $P \to M$ and an isomorphism $\eta \cong (P \times_{Spin(n)} R^n \to M).$ A spin manifold whose tangent bundle has a spin structure. Spin structures come up in differential geometry and index theory.

The stabilization map $SO(n) \to SO(n+1)$ induces a map $Spin(n) \to Spin(n+1).$ Thus a principle $Spin(n)-$ bundle $P \to M$ induces a principle $Spin(n+1)-$ bundle $P \times_{Spin(n)} Spin(n+1) \to M,$ and hence a spin structure on η gives a spin structure on $\eta \oplus \Sigma.$ A spin structure is a stable property.

A framing on a bundle gives a spin structure. A spin structure on a bundle gives an orientation. It turns out that a spin structure is equivalent to a framing on the 2-skeleton of M.

8.2.5. Stable Complex structure: An complex structure on a bundle η is a bundle map $J : \eta \to \eta$ so that $J \circ J = -Id.$ This forces the (real) dimension of η to be even. Equivalently complex structure is a reduction of the structure group to $GL(k,C) \subset GL(2k,R).$ The tangent bundle of a complex manifold admits a complex structure. One calls a manifold with a complex structure on its tangent bundle an almost complex manifold and it may ot may not admit the structure of a complex manifold. (It can be shown that S^6 is an almost complex manifold, but whether or not S^6 is a complex manifold is still an open question.)

One way to define a stable complex structure on a bundle η is as a section

$$J \Sigma \Gamma\left(Hom\left(\eta \oplus \Sigma^k, \eta \oplus \Sigma^k\right)\right)$$

Satisfying $J^2 = -Id$ in each fiber. Given such a J, one can extend it canonically to

$$\hat{J} = J \oplus i \in \Gamma\left(Hom\left(\eta \oplus \Sigma^k \oplus \Sigma^{2\ell}, \eta \oplus \Sigma^k \oplus \Sigma^{2\ell}\right)\right)$$

By identifying $\Sigma^{2\ell}$ with $M \times C^\ell$ and using multiplication by I to define $i\Sigma\Gamma\left(Hom\left(M \times C^\ell, M \times C^\ell\right)\right)$. As usual, two such structures are identified if they are homotopic. Note that odd-dimensional manifolds cannot have almost complex structures but may have stable almost complex structures.

If $\gamma : \eta \oplus \Sigma^k \cong \Sigma^\ell$ is a stable framing, up to equivalence we may assume that ℓ is even. Then identifying Σ^ℓ with $M \times C^{\ell/2}$ induces an stable complex structure on $\eta \oplus \Sigma^k$. Thus stably framed bundles have a stable complex structure.

Similarly, a complex structure determines an orientation, since a complex vector space has a canonical (real) orientation. To see this, notice that if $\{e_1 ...., e_r\}$ is a complex basis for a complex vector space, then $\{e_1, \mathrm{ie}_1, ..., \mathrm{e}_r, ie_r\}$ is a real basis whose orientation class is independent of the choice of the basis $\{e_1 ...., e_r\}$.

The orthogonal group $O(n)$ is a strong deformation retract of the general linear group $GL(n, R)$; this can be shown using the Gram-Schmidt process. This leads to a one-to-one correspondence between isomorphism classes of vector bundles and isomorphism classes of vector bundles and isomorphism classes of $R^n -$ bundles with structure group $O(n)$ over a paracompact base space. An $R^n -$ bundle with a metric has structure group $O(n)$. Conversely and $R^n -$ bundle with structure group $O(n)$ over a connected base space admits a metric, uniquely defined up to scaling. Henceforth in this chapter all bundles will have metrics with orthogonal structure group.

The following exercise indicates how to define a structure on a stable bundle in general.

Exercise 1:. Let $G = \{G_n\}$ be a sequence of topological groups with continuous homomorphisms $G_n \to G_{n+1}$ *and* $G_n \to O(n)$ so that the diagram

$$\begin{array}{ccc} G_n & \rightarrow & G_{n+1} \\ \downarrow & & \downarrow \\ O(n) & \hookrightarrow & O(n+1) \end{array}$$

Commutes for each n, where the injection $O(n) \to O(n+1)$ is defined by

$$A \mapsto \begin{bmatrix} A & 0 \\ 0 & 1 \end{bmatrix}.$$

Use this to define a stable G-structure on a bundle η. (Hint: either use classifying spaces or else consider the overlap functions for the stable bundle.)

Define what a homomorphism $G \to G'$ should be in such a way that a bundle with a stable G-structure becomes a bundle with a stable $G' -$ structure.

There are many examples of G-structures. As a perhaps unusual example, one could take G_n to be $O(n)$ *or* $SO(n)$ with the discrete topology. This spectrum arises in the study of flat bundles and algebraic K-theory.

For our previous examples, a framing corresponds to $G_n = 1,$ the trivial group for all n. The empty structure corresponds to $G_n = O(n).$ An orientation corresponds to $G_n = Spin(n) \to SO(n).$ An stable complex structure corresponds to $G_n = U([n/2]) \subset O(n).$

Concepts such as orientation and almost complex structure are more natural on the tangent bundle, while the Pontrjagin-Thom construction and hence bordism naturally deals with the stable normal bundle. The following exercise generalizes Theorem 8.13 and shows that in some cases one can translate back and forth.

Exercise:

1. Show that an orientation on the stable tangent bundle of a manifold determines one on the stable normal bundle and conversely.
2. Show that a complex structure on the stable tangent bundle of a manifold determines one on the stable normal bundle and conversely.

(Hint/discussion: The real point is that the tangent bundle and normal bundle are (stably) Whitney sum inverses, so one may as well consider bundles $\alpha \; and \; \beta$ over a finite-dimensional base space with a framing of $\alpha \oplus \beta.$ A complex structure on α is classified a map to $G_n(C^k)$ and β is equivalent to the pullback of the orthogonal complement of canonical bundle over the complex grassmannian, and hence V is equipped with a complex structure. Part 1 could be done using exterior powers or using the grassmannian of oriented n-planes in R^k.)

Definition: Given a G-structure, define the n-th G-bordism group of a space X to be the G-bordism classes of n-dimensional closed manifolds mapping to X with stable G-structures on the normal bundle of an embedding of the manifold in a sphere. Denote this abelian group (with disjoint union as the group operation)by

$$\Omega_n^G(X).$$

Notes

Thus an element of $\Omega_n^G(X)$ is represented by an embedded closed submanifold $M^n \subset S^k$, a continuous map $f: M \to X$, and a stable G-structure γ on the normal bundle γ Bordism is the equivalence relation generated by replacing k by k+1, and by

$$\left(M_0 \subset S^k, f_0, \gamma_0\right) \square \left(M_1 \subset S^k, f_1, \gamma_1\right)$$

Provided that there exists a compact manifold $W \subset S^k \times I$ with boundary

$M_0 \times \{0\} \cup M_1 \times \{1\}$ (which we identify with $M_0 II M_1$), a map $F: W \to X$ and a stable G-structure Γ on $v(W \to S^k \times I)$ which restricts to $(M_0 II M_1, f_0 II f_1, \gamma_0 II \gamma_1)$.

We previously used the notation Ω^{fr} for framed bordism, i.e $\Omega^{fr} = \Omega^1$ where $1 = G = \{G_n,\}$ the trivial group for all n,

We next want to associate spectra to bordism theories based on a stable structure. We have already seen how this works for framed bordism:

$$\Omega_n^{fr}(X) = \pi_n^S(X_+) = \lim_{\ell \to \infty} \pi_{n+\ell}\left(X_+ \wedge S^\ell\right)$$

i.e. framed bordism corresponds to the sphere spectrum $S = \{S^n, k_n\}$.

What do the other bordism theories correspond to? Does there exist a spectrum K for each structure G so that

$$\Omega_n^G(X) = H_n(X, K) = \lim_{\ell \to \infty} \pi_{n+\ell}\left(X_+ \wedge K_\ell\right)?$$

The answer is yes; the spectra for bordism theories are called Thom spectra MG. In particular, one can define G-cobordism by taking

$$H^n(X; MG) = \lim_{\ell \to \infty}\left[S^\ell X_+; MG_{n+\ell}\right]0.$$

We are using the algebraic topology terminology where cobordism is the theory dual (in the Spanier-Whitehead sense) to bordism. It is traditional for geometric topologists to call bordant manifolds "cobordant," but we will avoid this terminology in this book.

Thus we know that M1 is the sphere spectrum. We will give a construction for MG for any structure G.

8.3 CLASSIFYING SPACES

The construction of Thom spectral is accomplished most easily via the theory of classifying spaces. The basic result about classifying spaces is the following. The construction and the proof of this theorem is one of the student projects for Chapter 4.

Theorem 8.1. Given any topological group G, there exists a principal G-bundle $EG \to BG$ where EG is a contractible space. The construction is

Functorial, so that any continuous group homomorphism $\alpha : G \to H$ induces a bundle map

$$\begin{tikzcd} EG \arrow[r, "E\alpha"] \arrow[d] & EH \arrow[d] \\ BG \arrow[r, "B\alpha"] & BH \end{tikzcd}$$

Compatible with the actions, so that if $x \in EG, g \in G,$

$$E\alpha(x.g) = (E\alpha(x)).\alpha(g)$$

The space BG is called a classifying space for G.

The function

$$\Phi : M_{aps}(B, BG) \to \{\Pr incipal\, G - bundles\ over\, B\}$$

Defined by pulling back $(so\Phi(f) = f^*(EG))$ induces a bijection from the homotopy set $[B, BG]$ to the set of isomorphism classes of principal G-bundles over B, when B is a CW-complex (or more generally a paracompact space).

The long exact sequence for the fibration $G \to EG \to BG$ shows that $\pi_n BG = \pi_{n-1}G$. In fact, ΩBG is (weakly) homotopy equivalent to G, as one can see by taking the extended fiber sequence….
$\to \Omega EG \to \Omega Bg \to G \to EG \to BG$, computing with homotopy groups, and observing that EG and ΩEG are contractible. Thus the space BG is a delooping of G.

The following lemma is extremely useful.

Lemma 8.1: Let P: $E \to B$ be a principal G-bundle, and let $f : B \to BG$ be the classifying map. Then the homotopy fiber of f is weakly homotopy equivalent to E.

Proof. Turn $f : B \to BG$ into a fibration $q : B^{'} \to BG$ using Theorem 6.18 and let $F^{'}$ denote the homotopy fiber of $q : B^{'} \to BG$. Thus there is a commutative diagram.

$$\begin{tikzcd} B \arrow[rr, "h"] \arrow[dr, "f"'] & & B' \arrow[dl, "q"] \\ & BG & \end{tikzcd}$$

With h a homotopy equivalence. The fact that f is the classifying map for $p : E \to B$ implies that there is a commutative diagram

$$\begin{tikzcd} E \arrow[r, "\tilde{f}"] \arrow[d, "p"'] & EG \arrow[d] \\ B \arrow[r, "f"] & BG \end{tikzcd}$$

And since EG is contaractible, $f \circ p = q \circ h \circ p : E \to BG$ is nullhomotopic. By the homotopy lifiting property for the fibration $q : B^{'} \to BG$ it follows that $h \circ p : E \to B^{'}$ is homotopic into the fiber $F^{'}$ of $q : B^{'} \to BG$ and so one obtains a homotopy commutative diagram of spaces

$$\begin{tikzcd} E \arrow[r] \arrow[d, "p"'] & F' \arrow[d] \\ B \arrow[r, "h"] \arrow[d, "f"'] & B' \arrow[d, "q"] \\ BG \arrow[r, "="] & BG. \end{tikzcd}$$

The left edge is a fibration, h is a homotopy equivalence, and by the five lemma the map $\pi_n(E) \to \pi_n(F')$ is an isomorphism for all n.

In Lemma 8.23 one can usually conclude that the homotopy fiber of $f: B \to BG$ is in fact a homotopy equivalence. This would follow if we know that B' is homotopy equivalent to a CW-complex. This follows for most G by a theorem of Milnor [24].

Exercise : Show that given a principal G-bundle $E \to B$ there is fibration

$$\begin{tikzcd} E \arrow[r] & EG \times_G E \arrow[d] \\ & BG \end{tikzcd}$$

Where $EG \times_G E$ denotes the Borel construction.

8.4 CONSTRUCTION OF THE THOM SPECTRA

We proceed with the construction of the Thom spectra. We begin with a few preliminary notions.

Definition 8.2 If $E \to B$ is any vector bundle over a CW-complex B with metric then the Thom space of $E \to B$ is the quotient $D(E)/S(E)$, where $D(E)$ denotes the unit disk bundle of E and $S(E) \subset D(E)$ denotes the unit sphere bundle of E.

Notes

Notice that the zero section $B \to E$ defines an embedding of B into the Thom space.

The first part of the following exercise is virtually a tautology, but it is key to understanding why the spectra for bordism are given by Thom spaces.

Exercise.

1. If $E \to B$ is a smooth vector bundle over a smooth compact manifold B, then the thom space of E is a smooth manifold away from one point and the 0-section embedding of B into the Thom spce is a smooth embedding with normal bundle isomorphic to the bundle $E \to B$

2. The Thom space of a vector bundle over a compact base is homeomorphic to the one-point compactification of the total space.

Now let a G-structure be given. Recall that this means we have a sequence of continuous groups G_n and homomorphisms $G_n \to 0(n)$ and $G_n \to G_{n+1}$ such that the diagram

$$\begin{array}{ccc} G_n & \to & G_{n+1} \\ \downarrow & & \downarrow \\ O(n) & \hookrightarrow & O(n+1) \end{array}$$

Commutes,

We will construct the Thom spectrum for this structure form the Thom spaces of vector bundles associated to the principal bundles $G_n \to EG_n \to BG_{n.}$

Composing the bomomorphism $G_n \to 0(n)$ with the standard action of $0(n)$ on R^n defines an action of $G_n \text{on} R^n$. Use this action to form the universal R^n – vector bundle over BG_n

$$
\begin{tikzcd}
EG_n \times_{G_n} \mathbf{R}^n \arrow[d] \\
BG_n
\end{tikzcd}
$$

Let us denote this vector bundle by $V_n \to BG_n$. Notice that by our assumption that G_n maps to $o(n)$, this vector bundle has a metric, and so the unit sphere and disk bundles are defined.

Functoriality gives vector bundle maps (which are linear injection on fibers).

$$
\begin{tikzcd}
V_n \arrow[r] \arrow[d] & V_{n+1} \arrow[d] \\
BG_n \arrow[r] & BG_{n+1}
\end{tikzcd}
$$

Let MG_n denote the Thom space of $V_n \to BG_n$. Thus MG_n is obtained by collapsing the unit sphere bundle of V_n in the unit disk bundle to a point.

Lemma 8.25:

1. If $E \to B$ is a vector bundle, then the Thom space of $E \oplus \varepsilon$ is the reduced suspension of the Thom space of E.

2. A vector bundle map

$$
\begin{tikzcd}
E \arrow[r] \arrow[d] & E' \arrow[d] \\
B \arrow[r] & B'
\end{tikzcd}
$$

Which is an isomorphism preserving the metric on each fiber induces a map of Thom spaces.

Proof: To see why the first statement is true, note that an $O(n)$-equivariant homeomorphism $D^{n+1} \to D^n \times I$ determines an homeomorphism of $D(E \oplus \varepsilon)$ with $D(E) \times I$ which induces a homeomorphism $D(E \oplus \varepsilon)/S(E \oplus \varepsilon)$ with

$$\left(D(E) \times I\right)/\left(S(E) \times I\right) \cup d(E) \times \{0,1\}).$$

But it is easy to see that the this identification space is the same as the (reduced) suspension of $D(E)/S(E)$

The second statement is clear.

The following theorem states that the collection $MG = \{MG_n\}$ forms a spectrum. And the and that the corresponding homology theory is the bordism theory defined by the corresponding structure.

Theorem 8.26 : The fiberwise injection $V_n \to V_{n+1}$ extends to a (metric preserving) bundle map $V_n \oplus \varepsilon \to V_{n+1}$ which is an isomorphism on each fiber, and hence defines a map

$K_n : SMG_n \to MG_{n+1}$

Thus $\{MG_n, k_n\} = MG$ is a spectrum, called the Thom spectrum.

Moreover, the bordism groups $\Omega_n^G(X)$ are isomorphic to $H_n(X; MG)$.

Proof: Since the diagram

$$\begin{array}{ccc} G_n & \rightarrow & G_{n+1} \\ \downarrow & & \downarrow \\ O(n) & \hookrightarrow & O(n+1) \end{array}$$

Commutes, where $0(n) \to 0(n+1)$ the homomorphism is

$$A \mapsto \begin{bmatrix} A & 0 \\ 0 & 1 \end{bmatrix},$$

It follows by the construction of V_n that the pullback of V_{n+1} by the map $\gamma_n : BG_n \to BG_{n+1}$ splits canonically into a direct sum $\gamma_n^*(V_{n+1}) = V_n \oplus \varepsilon$.

Thus the diagram

$$\begin{tikzcd} V_n \arrow[r] \arrow[d] & V_{n+1} \arrow[d] \\ BG_n \arrow[r] & BG_{n+1} \end{tikzcd}$$

Extends to a diagram

$$\begin{tikzcd} V_n \oplus \varepsilon \arrow[r] \arrow[d] & V_{n+1} \arrow[d] \\ BG_n \arrow[r] & BG_{n+1} \end{tikzcd}$$

Which is an isomorphism on each fiber; this isomorphism preserves the metrics since the actions are orthogonal.

By Lemma 8.25, the above bundle map defines defines a map

$$k_n : SMG_n \to MG_{n+1}$$

Establishing the first part of the theorem.

We now outline how to establish the isomorphism

$$\Omega_n^G(X) = \lim_{l\to\infty} \pi_{n+\ell}(X_+ \wedge MG_\ell).$$

This is a slightly more complicated version of the Pontrjagin-Thom

construction we described before, using the basic property of classifying spaces.

We will first define the collapse map

$$c : \Omega_n^G(X) = \lim_{l\to\infty} \pi_{n+\ell}(X_+ \wedge MG_\ell).$$

Suppose $[W, f, \gamma] \in \Omega_n^G(X)$. So W is an n-manifold with G-Structure on its stable normal bundle, and $f : W \to X$ is a continuous map. Embed W in $S^{n+\ell}$ form some large ℓ so that the normal bundle $_{v}(W)$ has a G_ℓ -structure.

Let $F \to W$

$$\begin{tikzcd} P \arrow[r] \arrow[dr] & F \arrow[d] \\ & W \end{tikzcd}$$

Let $F \to W$ be the principal $O(\ell)$-bundle of orho normal frames in $v(W)$. The statement that $v(W)$. has a G_ℓ $-$structure is equivalent to saying that there is a principal $G_\ell - \text{bundle}\, P \to W$ and a bundle map

Which is equivalent with respect to the homomorphism

$$G_\ell \to O(\ell)$$

Let $c_1 : W \to BG_\ell$ classify the principal bundle $P \to W.$ then by definition $v(W)$ is isomorphic to the pullback $c_1^*(V_\ell)$.

Let U be a tubular neighborhood of W in $S^{n+\ell}$ and $D \subset U \subset S^{n+\ell}$

Correspond to the disk bundle. Define a map

$$h : S^{n+\ell} \to MG_\ell$$

By taking everything outside of D to the base point, and on D, take the composite

$$D \cong D(v(W)) \to D(V_\ell) \to MG_\ell$$

The product

$$f \times h : S^{n+\ell} \to X \times MG_\ell$$

Composes with the collapse

$$X \times MG_\ell \to X_+ \wedge MG_\ell$$

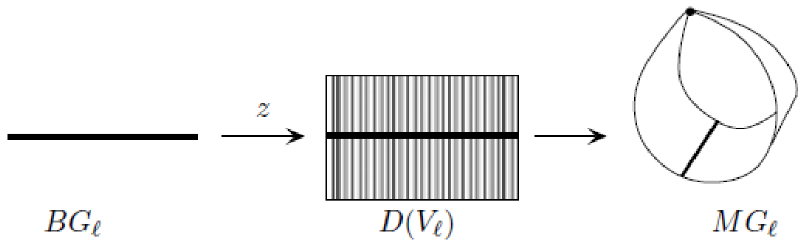
To give a map

$$\alpha = f \wedge h : S^{n+\ell} \to X_+ \wedge MG_{\ell.}$$

We have thus defined the collapse map

$$c : \Omega_n^G(X) \to \lim_{\ell\to\infty} \pi_{n+\ell}(X_+ \wedge MG_\ell) = H_n(X;MG)$$

To motivate the definition of the inverse of c, we will make a few comments on the above construction. The figure below illustrates that the composite of the zero section $z : BG_\ell \to D(V_\ell)$ and the quotient map $D(V_\ell) \to MG_\ell$ is a embedding.



We thus will coincide BG_ℓ to be a subset of MG_ℓ. Then in the above construction of the collapse map c, $W = \alpha^{-1}(X \times BG_\ell)$.

Next we use transversality to define the inverse of this the collapse map c. Represent

$$\hat{\alpha} \in H_n(X;MG) \text{ by}$$

$$\alpha : S^{n+\ell} \to X_+ \wedge MG_{\ell.}$$

Observe that the composite

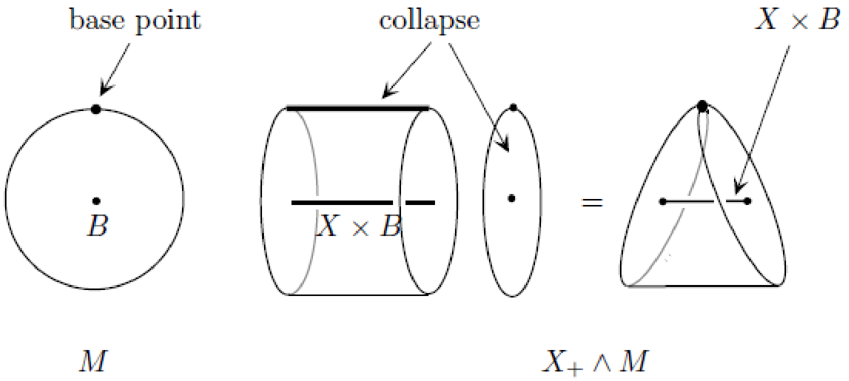
$$X \times BG_\ell \to X_+ \times MG_\ell \to X_+ \wedge MG_\ell$$

Is an embedding, since:

1. BG_ℓ misses the base point of MG_ℓ, and

2. The base point of X_+ misses X.

(The following figure gives an analogue by illustrating the embedding of $X \times B$ in $X_+ \wedge M$ if B is a point , M is a D^2 – bundle over B, and X is a interval.)



Furthermore $$X \times BG_\ell \subset X_+ \wedge MG_\ell$$

Has a neighborhood which is isomorphic to the pullback $\pi_2^* V_\ell$ where $\pi_2 : X \times BG_\ell \to BG_\ell$ is the projection on the second factor.

Transversality, adapted to this setting, says that $\alpha : S^{n+\ell} \to X_+ \wedge MG_\ell$ is a smooth manifold, and whose tubular neighborhood, i.e. the normal bundle of W, has a G-structure. The composite of $\beta : W \to X \times BG_\ell$ and $\Pr_1 : X \times BG_\ell \to X$ give the desired element

$$(W \to X) \in \Omega_n^G(X).$$

We sort of rushed through the construction of the inverse map to c, so we will backtrack and discuss some details. For every point in $BG_{\ell,}$ there

is a neighborhood $U \subset BG_\ell$ over which the bundle $V_\ell \to BG_\ell$ is trivial and so there is a map

$$\alpha^{-1}(X \times U) \to D^\ell / S^\ell$$

Defined by composing α with projection on the fiber. Transversality then applies to this map between manifolds and one can patch together to get β using partitions of unity. Furthermore, transversality gives a diagram of bundle maps, isomorphisms in each fiber,

$$\begin{tikzcd}
\nu(W \hookrightarrow S^{n+\ell}) \arrow[r] \arrow[d] & X \times V_\ell \arrow[r] \arrow[d] & V_\ell \arrow[d] \\
W \arrow[r, "\beta"] & X \times BG_\ell \arrow[r, "pr_2"] & BG_\ell
\end{tikzcd}$$

So that the normal bundle of W inherits a G-structure.

Next note that replacing ℓ by $\ell+1$ leads to the same bordism element.

Stabilizing the normal bundle

$$v\left(W \to S^{n+\ell}\right) \to v\left(W \to S^{n+\ell}\right) \oplus \varepsilon = v\left(W \to S^{n+\ell+1}\right)$$

Corresponds to including $W \subset S^{n+\ell} \subset S^{n+\ell+1}$.

Since the composite

$$SS^{n+\ell} \xrightarrow{Sf} S\left(X_+ \wedge MG_\ell\right) \xrightarrow{k_\ell \wedge Id} X_+ \wedge MG_{\ell+1}$$

Replaces the tubular neighborhood of

$X \times BG_\ell$, i.e $X \times V_\ell$ by $X \times \left(V_\ell \oplus \varepsilon\right)$, the construction gives a well-defined stable G-structure on the stable normal bundle of W.

The full proof that the indicated map $H_n\left(X : MG\right) \to \Omega_n^G\left(X\right)$ is well-defined and is the inverse of c is a careful but routine check of details involving bordisms, homotopies, and stabilization.

Taking X to be a point, we see that the groups (called the coefficients) $\Omega_n^G = \Omega_n^G\left(pt\right)$ are isomorphic to the homotopy groups $\lim_{\ell \to \infty} \pi_{n+\ell}\left(MG_\ell\right)$, since $pt_+ \wedge M = M$.

As an example of how these coefficients can be understood geometrically, consider oriented bordism, corresponding to $G_n = SO\left(n\right)$. the coefficients Ω_n^{SO} equal $\pi_{n+\ell}\left(MSO_\ell\right)$ for ℓ large enough. Some basic computations are the following.

Notes

1. An oriented closed 0-manifold is just a signed finite number of points. This bounds a 1-manifold if and only if the sum of the signs is zero. Hence $\Omega_n^{SO} \cong Z$. Also, $\pi_\ell MSO_\ell = Z$ for $\ell \geq 2$.

2. Every oriented closed 1-manifold bounds an oriented 2-manifold, since $S^1 = \partial D^2$. Therefore $\Omega_1^{SO} = 0$.

3. Every oriented 2-manifold bounds an oriented 3-manifold since any oriented 2-manifold 2-manifold embeds in R^3 with one of the two complementary components compact. Thus $\Omega_2^{SO} = 0$.

4. A theorem of Rohlin states that every oriented 3-manifold bounds a 4-manifold. Thus $\Omega_3^{SO} = 0$.

5. An oriented 4-manifold has a signature in Z, i.e. the signature of its intersection form. A good exercise using Poincare duality (see the projects for Chapter 3) shows that this is an oriented bordism invariant, and hence defines a homomorphism $\Omega_4^{SO} \to Z$. This turns out to be an isomorphism. More generally the signature defines a out to be an isomorphism. More generally the signature defines a map $\Omega_{4k}^{SO} \to Z$ for k. This is a surjection since the signature of CP^{2k} is 1.

6. It is a fact that away from multiples of 4, the oriented bordism groups are torsion, i.e. $\Omega_n^{SO} \otimes Q = 0$ if $n \neq 4k$.

7. For all n, Ω_n^{SO} is finitely generated, in fact, a finite direct, sum of Z's and Z/2's.

Statements 5,6, and 7 can be proven by computing $\pi_{n+\ell}\left(MSO_\ell\right)$. How does one do this? A starting point is the Thom isomorphism theorem, which says that for all k,

$$H_n\left(BSO(\ell) \cong \tilde{\tilde{H}}_{n+\ell}\left(MSO_\ell\right)\right)$$

(Where $\tilde{\tilde{H}}$ denotes reduced cohomology). The cohomology of $BSO(n)$ can be studied in several ways, an so one can obtain information about the cohomology of MSO_ℓ by this with the Hurewicz theorem and other methods leads ultimately to a complete computation of oriented bordism (due to C.T.C. Wall), and this technique was generalized by Adams to a machine called the Adams spectral sequence. We will return to the Thom isomorphism theorem in Chapter 10.

One the coefficients are understood, one can use the fact bordism is a homology theory to compute $\Omega_n^{SO}(X)$. For now we just remark that there is a map $\Omega_n^{SO}(X) \to H_n(X)$ defined by taking $f: M \to X$ to the image of the fundamental class $f*[M]$. Thus for example, the identity map on a closed, oriented manifold M^n is non-zero in $\Omega_n^{SO}(M)$.

We can also make an elementary remark about unoriented bordism, which corresponds to $G_n = O(n)$. Notice first that for any $\alpha \in \Omega_n^O(X)$, $2\alpha = 0$. Indeed, if $f: V^n \to X$ represents α, take $F: V \times I \to X$ to be $F(x,t) = f(x) \text{ then } \partial(V \times I, F) = 2(V,f)$. Thus $\Omega_n^O(X)$ consists only of elements of order 2. The full computation of unoriented bordism is due to Thom. We will discuss this more in Section 10.10

Exercise: Show that $\Omega_0^O = Z/2, \Omega_1^O = 0 \text{ and } \Omega_2^O = Z/2$.

(Hint: For Ω_2^O use the classification theorem for closed surfaces, then show that if a surface F is a boundary of a 3-manifold, then dim $H^1(F:Z/2)$ is even.)

There are several conversations regarding notation for bordism groups; each has its advantages. Given a structure defined by a sequence $G:\{G_n\}$, one can use the notation

$$\Omega_*^G(X), H_*(X;MG) \text{ or } MG_*(X)$$

There is a generalization of a G-structure called a B-structure. It is given by a sequence of commutative diagrams

$$\begin{tikzcd} B_n \arrow[r] \arrow[d, "\xi_n"'] & B_n \arrow[d, "\xi_{n+1}"] \\ BO_n \arrow[r] & BO_{n+1} \end{tikzcd}$$

Where the vertical maps are fib rations. A G-structure in the old sense gives a $BG = \{BG_n\}$-structure. A B-structure has a Thom spectrum TB=$\{T(\xi_n)\}$, where ξ_n here denotes the vector bundle pulled back from the canonical bundle over $BO_{n.}$ There is a notion of a stable B-structure on a normal bundle of an embedded M, which implies that there is a map from the (stablized) normal bundle to ξ_k. There is a Pontrjagin-Thom isomorphism

$$\Omega_n^B(X) \cong H_n(X;TB).$$

8.5 GENERALIZED HOMOLOGY THEORIES

We have several factors from (based) spaces to graded abelian groups: Stable homotopy $\pi_n^S(X)$, bordism $\Omega_n^G(X)$, or, more generally, homology of a space with coefficients in a spectrum $H_n(X:K)$. These are examples of generalized homology theories. Generalized homology theories come in two (equivalent) flavors, reduced and unreduced. Unreduced theories apply to unbased spaces and pairs. Reduced theories are functors on based spaces. The equivalence between the two points of view is obtained by passing from (X, A) to X / A and from X to X_+.

There are three high points to look out for in our discussion of homology theories.

* The axioms of a (co) homology theory are designed for computations.

One fist computes the coefficients of the theory.

Check Your Progress

1. Prove: Given any topological group G, there exists a principal G-bundle $EG \to BG$ where EG is a contractible space.

2. Prove: Let P: $E \to B$ be a principal G-bundle, and let $f : B \to BG$ be the classifying map. Then the homotopy fiber of f is weakly homotopy equivalent to E.

3. Prove: If $E \to B$ is a vector bundle, then the Thom space of $E \oplus \varepsilon$ is the reduced suspension of the Thom space of E.

8.6 LET US SUM UP

1. Given a G-structure, define the n-th G-bordism group of a space X to be the G-bordism classes of n-dimensional closed manifolds mapping to X with stable G-structures on the normal bundle of an embedding of the manifold in a sphere. Denote this abelian group (with disjoint union as the group operation)by $\Omega_n^G(X)$.
2. $E \to B$ be a principal G-bundle, and let $f : B \to BG$ be the classifying map. Then the homotopy fiber of f is weakly homotopy equivalent to E.
3. If $E \to B$ is any vector bundle over a CW-complex B with metric then the Thom space of $E \to B$is the quotient $D(E)/S(E)$, where $D(E)$ denotes the unit disk bundle of E and $S(E) \subset D(E)$ denotes the unit sphere bundle of E.

8.8 KEY WORDS

Homotopy

Isomorphism

Homomorphism

General Bordism

Homology

8.9 QUESTIONS FOR REVIEW

1. Explain about general bordism theories
2. Explain about generalized homology theories

8.19 SUGGESTIVE READINGS AND REFERENCES

1. Algebraic Topology – Satya Deo

2. Lectures notes in Algebraic Topology- James F. Davis Paul Kirk

3. Introduction to Algebraic Topology and Algebraic Geometry- U. Bruzzo

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8.10 ANSWER TO CHECK YOUR PROGRESS

1. See section 8.3
2. See section 8.3
3. See section 8.4

UNIT- 9 SPECTRAL SEQUENCES

STRUCTURE

9.0 Objective

9.1 Introduction

9.2 Definition of a spectral sequence

9.3 The Leray-Serre-Atiyah-Hirzebruch spectral sequence

9.4The edge homomorphisms and the transgressions

9.5 Let us sum up

9.6 Key words

9.7 Questions for Review

9.8 Suggested readings and references

9.9 Answers to check your progress questions

9.0 OBJECTIVE

In this unit we will learn and understand about definition of Spectral sequence, The Leray-Serre-Atiyah-Hirzebruch spectral sequence and The edge homomorphism’s and the transgression.

9.1 INTRODUCTION

Spectral sequences are powerful computational tools in topology. They also can give quick proofs of important theoretical results such as the Hurewicz theorem and the Freudential suspension theorem. Computing with spectral sequences is somewhat like computing integrals in calculus; it is helpful to have ingenuity and a supply of tricks, and even so, you may not arrive at the final solution to your problem. There are many spectral sequences which give different kinds of information. We will focus on one important spectral sequence, the Leray-Serre-Atiyah-Hirzebruch spectral sequence which takes as input a fibration over a CW-complex and a generalized homology or cohomology theory. This spectral sequence exhibits a complicated relationship between the generalized (co)homology of the total space and fiber and the ordinary (co)homology of the base. Many other spectral sequences can be derived from this one by judicious choice of fibration and generalized (co)homology theory.

Carefully setting up and proving the basic result requires very careful bookkeeping; the emphasis in these notes will be on applications and how to calculate.

9.2 DEFINITION OF A SPECTRAL SEQUENCE

Definition 9.1. A spectral sequence is a homological object of the following type:

One is given a sequence of chain complexes

$$\left(E^r, d^r\right) \text{for r = 1, 2, . . .}$$

and isomorphisms:

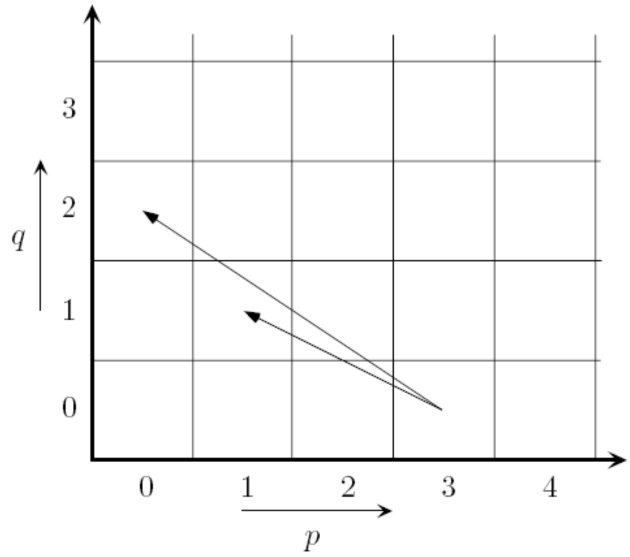
$$E^{r+1} \cong H\left(E^r, d^r\right) = \frac{\ker d^r : E^r \to E^r}{\operatorname{Im} d^r : E^r \to E^r}.$$

The isomorphisms are fixed as part of the structure of the spectral sequence so henceforth we will fudge the distinction between "$\cong$" and "$=$" in the above context.

In this definition the term "chain complex" just means an abelian group (or R-module) with an endomorphism whose square is zero. In many important contexts, the spectral sequence has more structure, namely the chain complexes E^r are graded or even bigraded, that is, E^r decomposes as a direct sum of terms $E^r_{p,q}$ for $(p,q) \in Z \oplus Z.$ moreover the differentials d^r have a well-defined bidegree. For example, in a homology spectral sequence, usually d^r has bidegree $(-r, r-1)$. In other words $d^r\left(E^r_{p,q}\right) \subset E^r_{p-r,q+r-1}$.

A student first exposed to this plethora of notation may be intimidated; the important fact to keep in mind is that a bigrading decomposes a big object $\left(E^r\right)$ into bite-sized pieces $\left(E^r_{p,q}\right)$. Information about the $E^r_{p,q}$

for some pairs (p,q) gives information about $E^{r+1}_{p,q}$ for (probably fewer) pairs (p,q). But with luck one can derive valuable information. For example, from what has been said so far you should easily be able to see that if $E^r_{p,q}=0$ for some fixed pair (p,q), then $E^{r+k}_{p,q}=0$ for all $k\geq 0$. This simple observation can sometimes be used to derive highly non-trivial information. When computing with spectral sequences it is very useful to draw diagrams like the following.



In this picture the short arrow depicts the differential $d^2: E^2_{3,0} \to E^2_{1,1}$ and the long arrow corresponds to the differential $d^3: E^3_{3,0} \to E^3_{0,2}$

One usually computes with a spectral sequence in the following way. A theorem will state that there exists a spectral sequence so that:

1. the modules E^2 (or E^1) can be identified with something known, and
2. the limit

$$E^{\infty} = \lim_{r\to\infty} E^r$$

is related to something one wishes to compute.

It can also work the opposite way, E^{∞} can be related to something known and E^2 can be related to something we wish to compute. In either case, this gives a complicated relationship between two things. The relationship usually involves exact sequences. In favorable circumstances information can be derived by carefully analysing this relationship.

As an example to see how this may be used, the Leray–Serre spectral sequence of a fibration implies that if $F \to E \to B$ is a fibration with B simply connected, then there is a spectral sequence with

$$E^2_{p,q} \cong H_p(B;Q) \otimes H_q(F;Q)$$

And with

$$H_N(E;Q) \cong \otimes_P E^{\infty}_{p,n-p}.$$

This establishes a relationship between the homology of the base, total space, and fiber of a fibration. Of course, the hard work when computing with this spectral sequence is in getting from E^2 to E^{∞}. But partial computations and results are often accessible. For example, we will show later (and the reader may wish to show as an exercise now) that if $\oplus_p H_p(B;Q)$ and $\oplus_q H_q(F;Q)$ are finite-dimensional, then so is $\oplus_n H_n(E;Q)$ and

$$x(B).\ x(F) = x(E).$$

Another example: if $H_p(B;Q) \otimes H_{n-p}(F;Q) = 0$ for all p, then

$H_n(E;Q) = 0$. This generalizes a similar fact which can be proven for the trivial fibration

$B \times F \to B$ using the Kunneth theorem.

The next few definitions will provide us with a language to describe the way that the parts of the spectral sequence fit together.

Definition 9.2. A filtration of an R-module A is an increasing union

$$0 \subset \ldots\ldots \subset F_{-1} \subset F_0 \subset F_1 \subset \ldots\ldots \subset F_p \subset \ldots\ldots\ldots \subset A.$$

of submodules. A filtration is convergent if the union of the F_p's is A and their intersection is 0.

If A itself is graded, then the filtration is assumed to preserve the grading i.e. $F_p \cap A_n \subset F_{p+1} \cap A_n$. If A is graded, then we bigrade the filtration by setting

$$F_{p,q} = F_p \cap A_{p+q}.$$

We will mostly deal with filtrations that are bounded below, i.e. $F_s = 0$ for some s, or bounded above, i.e. $F_t = A$ for some t, or bounded, i.e.

bounded above and bounded below. In this book, we will always have $F_{-1} = 0$.

Definition 9.3. Given a filtration $F = \{F_n\}$ of an R-module A the associated graded module is the graded R-module denoted by Gr(A, F) and defined by

$$Gr(A,F)_p = \frac{F_p}{F_{p-1}}.$$

We will usually just write Gr(A) when the filtration is clear from context. In general, one is interested in the algebraic structure of A rather than Gr(A). Notice that Gr(A) contains some (but not necessarily all) information about A. For example, for a convergent filtration:

1. If Gr(A) = 0, then A = 0.
2. If R is a field and A is a finite dimensional vector space, then each F_i is a subspace and Gr(A) and A have the same dimension. Thus in this case Gr(A) determines A up to isomorphism. This holds for more general R if each $\mathrm{Gr}(A)_n$ is free and the filtration is bounded above.
3. If R = Z, then given a prime b, information about the b-primary part of Gr(A) gives information about the b-primary part of A; e.g. if $\mathrm{Gr}(A)_p$ has no b-torsion for all p then A has no b-torsion for all p. However, the b-primary part of Gr(A) does not determine the b-primary part of A; e.g. if $\mathrm{Gr(A)}_0 = \mathrm{Z}$, $\mathrm{Gr(A)}_1 = \mathrm{Z}/2$, and $\mathrm{Gr}(\mathrm{A})_n = 0$ for $n \neq 0,1$, it is impossible to determine whether $A \cong Z$ or $A \cong \mathrm{Z} \oplus Z/2$.

In short, knowing the quotients $Gr(A)_p = F_p / F_{p-1}$ determines A up to "extension questions," at least when the filtration is bounded.

Definition 9.4. A bigraded spectral sequence $\left(E_{p,q}^r, d^r\right)$ is called a homology

spectral sequence if the differential d^r has bidgree $(-r, r-1)$.

Definition 9.5. Given a bigraded homology spectral sequence $\left(E_{p,q}^r, d^r\right)$, and a graded R-module A_*, we say the spectral sequence converges to A_* and write

$$E^2_{p,q} \Rightarrow A_{p+q}$$

if:

1. for each p, q, there exists an r_0 so that $d^r_{p,q}$ is zero for each $r \geq r_0$ (by Exercise 145 below this implies $E^r_{p,q}$ surjects to $E^{r+1}_{p,q}$ for $r \geq 0$) and,

2. there is a convergent filtration of A_*, so that for each n, the limit $E^\infty_{p,n-p} = \lim_{r\to\infty} E^r_{p,n-p}$ is isomorphic to the associated graded module $Gr(A_*)_p$.

In many favorable situations (e.g. first-quadrant spectral sequences where $E^2_{p,q} = 0$ if p < 0 or q < 0) the convergence is stronger, namely for each pair (p, q) there exists an r_0 so that $E^r_{p,q} = E^\infty_{p,q}$ for all $r \leq r_0$.

An even stronger notion of convergence is the following. Suppose that there exists an r_0 so that for each (p, q) and all $r \geq r_0, E^r_{p,q} = E^\infty_{p,q}$. When this happens we say the spectral sequence collapses at E^{r0}.

Exercise. Fix $p, q \in Z \oplus Z$.

1. Show that if there exists $r_0(p,q)$ so that $d^r_{p,q} = 0$ for all $r \geq r_0(p,q)$, then there exists a surjection $E^r_{p,q} \to E^{r+1}_{p,q}$ for all $r \geq r_0(p,q)$.

2. Show that if $E^2_{p,q} = 0$ whenever p < 0 then there exists a number $r_0 = r_0(p,q)$ as above.

Theorems on spectral sequences usually take the form: "There exists a Spectral sequence with $E^2_{p,q}$ some known object converging to A_*." This is an abbreviated way to say that the E^∞-terms are on the one hand the limits of the E^r-terms, and on the other the graded pieces in the associated graded $Gr(A_*)$ to A_*.

9.3 THE LERAY-SERRE-ATIYAH-HIRZEBRUCH SPECTRAL SEQUENCE

Notes

Serre, based on earlier work of Leray, constructed a spectral sequence converging to $H_*(E)$, given a fibration

$$F \to E \xrightarrow{f} B.$$

Atiyah and Hirzebruch, based on earlier work of G. Whitehead, constructed a spectral sequence converging to $G_*(B)$ where G_* is an additive generalized homology theory and B is a CW-complex. The spectral sequence we present here is a combination of these spectral sequences and converges to $G_*(E)$ when G_* is an additive homology theory. The spectral sequence is carefully constructed .and we refer you there for a proof.

We may assume B is path connected by restricting to path components, but we do not wish to assume B is simply connected. In order to deal with this case we will have to use local coefficients derived from the fibration. Theorem 6.12 shows that the homotopy lifting property gives rise to a homomorphism $\pi_1 B \to$ {Homotopy classes of homotopy equivalences $F \to F$ }.

Applying the (homotopy) functor G_n one obtains a representation

$$\pi_1 B \to Aut(G_n F)$$

for each integer n. Thus for each $n, G_n(F)$ has the structure of a Z $[\pi_1 B]$ module or, equivalently, one has a system of local coefficients over B with fiber $G_n(F)$. (Of course, if $\pi_1 B = 1$ than this is a trivial local coefficient system.)

Taking ordinary) homology with local coefficients, we can associate the group $H_p(B; G_q F)$ to each pair of integers p, q. Notice that $H_p(B; G_q F)$ is zero if p < 0.

Theorem 9.6. Let $F \to E \xrightarrow{f} B$ be a fibration, with B a path connected CW-complex. Let G_* be an additive homology theory. Then there exists a spectral sequence

$$H_p(B; G_q F) \cong E^2_{p,q} \Rightarrow G_{p+q}(E).$$

Exercise. If G_* is an additive, isotropic homology theory, then the hypothesis that B is a CW-complex can be omitted. (Hint: for any space B there is a weak homotopy equivalence from a CW-complex to B.)

As a service to the reader, we will explicitly unravel the statement of the above theorem. There exists

1. A (bounded below) filtration

$$0 = F_{-1,n+1} \subset F_{0,n} \subset F_{1,n-1} \subset \ldots.. \subset F_{p,n-p} \subset \ldots.. \subset G_n(E)$$

of $G_n(E) = \cup_p F_{p,n-p}$ for each integer n.

2. A bigraded spectral sequence $\left(E^r_{*,*}, d^r\right)$ such that the differential d^r has bidegree $(-r, r-1)$ (i.e. $d^r\left(E^r_{p,q}\right) \subset E^r_{p-r,q+r-1}$), and so

$$E^{r+1}_{p,q} = \frac{\ker d^r : E^r_{p,q} \to E^r_{p-r,q+r-1}}{\operatorname{Im} d^r : E^r_{p+r,q-r+1} \to E^r_{p,q}}.$$

3. Isomorphisms $E^2_{p,q} \cong H_p\left(B; G_q F\right)$.

This spectral sequence converges to $G_*(E)$. That is, for each fixed p, q, there exists an r_0 so that

$$d^r : E^r_{p,q} \to E^r_{p-r,q+r-1}$$

is zero for all $r \geq r_0$ and so

$$E^{r+1}_{p,q} = E^r_{p,r} / d^r\left(E^r_{p+r,q-r+1}\right)$$

For all $r \geq r_0$.

Define $E^\infty_{p,q} = \lim_{r\to\infty} E^r_{p,q}$. There is an isomorphism

$$F_{p,q} / F_{p-1,q+1} \cong E^\infty_{p,q},$$

i.e.

$$Gr\left(G_n E\right)_p \cong E^\infty_{p,n-p}$$

with respect to the filtration of $G_n(E)$.

In this spectral sequence, some filtrations of the groups $G_n(E)$ are given, with the associated graded groups made up of the pieces $E^\infty_{p,n-p}$.

So, for example, if $G_n(E) = 0$, then $E^\infty_{p,n-p} = 0$ for each $p \in$ Z.

The filtration is given by

$$F_{p,n-p} = \mathrm{Im}\left(G_n\left(f^{-1}\left(B^p\right)\right) \to G_n E\right)$$

where $f : E \to B$ is the fibration and B^p denotes the p-skeleton of B.

As a first non-trivial example of computing with spectral sequences we consider the problem of computing the homology of the loop space of a sphere. Given $k > 1$, let $P = P_{x_0} S^k$ be the space of paths in S^k which start at $x_0 \in S^k$. As we saw in Chapter 6 evaluation at the endpoint defines a fibration $P \to S^k$ with fiber the loop space ΩS^k. Moreover the path space P is contractible. The spectral sequence for this fibration (using homology with integer Coefficients for G_*) has $E^2_{p,q} = H_p\left(S^k; H_q\left(\Omega S^k\right)\right)$. The coefficients are untwisted since $\pi_1\left(S^k\right) = 0$. Therefore

(9.1) $$E^2_{p,q} = \begin{cases} H_q\left(\Omega S^k\right) & \textit{if } p = 0 \textit{ or } p = k \\ 0 & \textit{otherwise.} \end{cases}$$

In particular this is a first-quadrant spectral sequence.

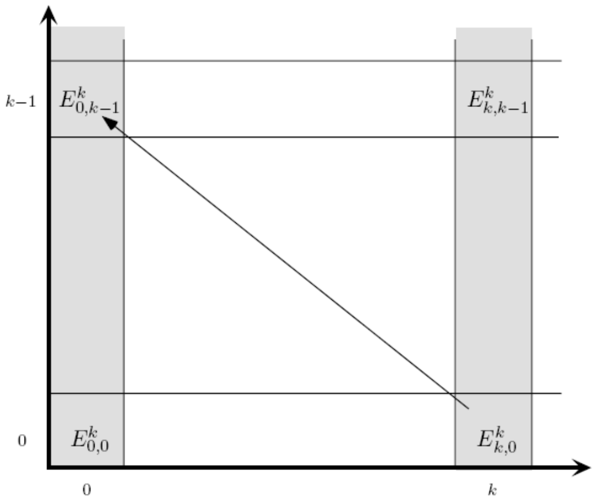
Since $H_n(P) = 0$ for all $n \neq 0$, the filtration of $H_n(P)$ is trivial for $n > 0$ and so $E^\infty_{p,q} = 0$ if $p + q > 0$. Since this is a first-quadrant spectral sequence, $E^\infty_{p,q} = 0$ for all $(p,q) \neq (0,0)$, and, furthermore, given any $(p,q) \neq (0,0)$, $E^\infty_{p,q} = 0$ for some r large enough.

Now here's the cool part. Looking at the figure and keeping in mind the fact that the bidegree of d^r is $(-r, r-1)$, we see that all differentials $d^r : E^r_{p,q} \to E^r_{p-r,q+r-1}$ either:

1. start or end at a zero group, or
2. $r = k$ and $(p,q) = (k,q)$ with $q \geq 0$, so that

$$d^k : E^k_{k,q} \to E^k_{0,q+k-1}.$$

The following picture shows the E^k-stage and the differential $d^k : E^k_{k,0} \to E^k_{0,k-1}$. The shaded columns contain the only possible non-zero entries, since $E^2_{p,q} = 0$ if $p \neq 0$ or k.



Hence

(9.2) $$E^2_{p,q} = E^3_{p,q} = \ldots\ldots = E^k_{p,q}.$$

Thus if $(p,q) \neq (0,0)$,

$$0 = E^{\infty}_{p,q} = E^{k+1}_{p,q} = \begin{cases} \ker d^k : E^k_{k,q} \to E^k_{0,q+k-1} & \text{if } (p,q) = (k,q) \\ Co\ker d^k : E^k_{k,q} \to E^k_{0,q+k-1} & \text{if } (p,q) = (0,q+k-1) \\ 0 & \text{Otherwise} \end{cases}$$

Therefore, the spectral sequence collapses at E^{k+1}.

Hence d^k is an isomorphism, i.e. $E^k_{k,q} \cong E^k_{0,q+k-1}$ whenever $(k,q) \neq (0,0)$ or $q \neq 1-k$. Using Equations (9.2) and (9.1) we can restate this as

$$H_q\left(\Omega S^k\right) \cong H_{q+k-1}\left(\Omega S^k\right).$$

Using induction, starting with $H_0\left(\Omega S^k\right) = 0$, we conclude that

(9.3) $$H_0\left(\Omega S^k\right) = \begin{cases} Z & \text{if } q = a(k-1), a \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Exercise : If $S^k \to S^l \xrightarrow{f} S^m$ is a fibration, then $l = 2m-1$ and $k = m-1$. (In fact, it is a result of Adams that there are only such fibrations for m = 1, 2, 4 and 9.)

Returning to our general discussion, notice that $E^{r+1}_{p,q}$ and $E^{\infty}_{p,q}$ are subquotients of $E^r_{p,q}$; in particular, since $E^2_{p,q} \cong H_p\left(B; G_q F\right)$ we conclude the Following fundamental fact.

Theorem 9.7. The associated graded module to the filtration of $G_n(E)$ has graded summands which are subquotients of $H_p\left(B; G_{n-p} F\right)$.

This fact is the starting point for many spectral sequence calculations. For example,

Theorem 9.8. If $H_p\left(B;G_{n-p}F\right)=0$ for all p, then $G_n\left(E\right)=0$.

Proof. Since $E^2_{p,n-p}=0$ for each p, it follows that $E^\infty_{p,n-p}=0$ for each p and so $G_n\left(E\right)=0$.

9.4. THE EDGE HOMOMORPHISMS AND THE TRANSGRESSION

Before we turn to more involved applications, it is useful to know several facts about the Leray-Serre-Atiyah-Hirzebruch spectral sequence. These facts serve to identify certain homomorphisms which arise in the guts of the spectral sequence with natural maps induced by the inclusion of the fiber or the projection to the base in the fibration.

Lemma 9.9. In the Leray-Serre-Atiyah-Hirzebruch spectral sequence there

is a surjection

$$E^2_{0,n}\to E^\infty_{0,n} \quad \text{for all n.}$$

Proof. Notice that

$$E^{r+1}_{0,n}=\frac{\ker d^r: E^r_{0,n}\to E^r_{r,n+r-1}}{\operatorname{Im} d^r: E^r_{r,n-r+1}\to E^r_{0,n}} \quad for \quad r>1.$$

But, since $E^2_{p,q}=0$ for $p<0$, we must have $E^2_{-r,q}=0$ for all q and so also its sub quotient $E^r_{-r,q}=0$ for all q.

Hence (ker $d^r: E^r_{0,n}\to E^r_{-r,n+r-1}$) = $E^r_{0,n}$ and so

$$E^{r+1}_{0,n}=\frac{E^r_{0,n}}{\operatorname{Im} d^r}.$$

Thus each $E^r_{0,n}$ surjects to $E^{r+1}_{0,n}$ and hence also to the limit $E^\infty_{0,n}$.

Proposition 5.14 says that that if V is any local coefficient system over a path connected space B, then

$$H_0\left(B;V\right)=V/\langle \upsilon-\alpha.\upsilon \mid \upsilon\in V, \alpha\in\pi_1 B\rangle.$$

Applying this to $V=G_n\left(F\right)$, it follows that there is a surjection

$$G_n\left(F\right)\to H_0\left(B;G_nF\right). \tag{9.4}$$

We can now use the spectral sequence to construct a homomorphism $G_*(F) \to G_*(E)$. Theorem 9.10 below asserts that this homomorphism is just the homomorphism induced by the inclusion of the fiber into the total space.

Since $F_{-1,n-1} = 0$, $E^\infty_{0,n} \cong F_{0,n} / F_{-1,n+1} F_{0,n} \subset G_n(E)$.

This inclusion can be pre composed with the surjections of Lemma 9.9 and Equation (9.4) to obtain a homomorphism (called an edge homomorphism)

(9.5) $$G_n(F) \to H_0(B; G_n F) \cong E^2_{0,n} \to E^\infty_{0,n} \subset G_n(E).$$

Theorem 9.10. The edge homomorphism given by (9.5) equals the map $i_* : G_n(F) \to G_n(E)$ induced by the inclusion $i : F \to E$ by the homology theory G_*..

Another simple application of the spectral sequence is to compute oriented bordism groups of a space in low dimensions. We apply the Leray-Serre-Atiyah-Hirzebruch spectral sequence to the fibration $pt \to X \xrightarrow{Id} X$, and take $G_* = \Omega^{SO}_*$, oriented bordism.

In this case the Leray-Serre-Atiyah-Hirzebruch spectral sequence says

$$H_p\left(X; \Omega^{SO}_q(pt)\right) \Rightarrow \Omega^{SO}_{p+q}(X).$$

Notice that the coefficients are untwisted; this is because the fibration is Trivial. Write $\Omega^{SO}_n = \Omega^{SO}_n(pt)$. Note that $pt \to X$ is split by the constant map, hence the edge homomorphism $\Omega^{SO}_n = \Omega^{SO}_n(X)$ is a split injection, so by Theorem 9.10, the differentials $d^r : E^r_{r,n-r+1} \to E^r_{0,n}$ whose targets are on the vertical edge of the first quadrant must be zero, i.e. every element of $E^2_{0,n}$ survives to $E^\infty_{0,n} = \Omega^{SO}_n$.

Recall from Section 9.7 that $\Omega^{SO}_n = 0$ for q = 1, 2, 3, and Ω^{SO}_q = Z for q = 0 and 4. Of course $\Omega^{SO}_q = 0$ for $q < 0$. Thus for $n = p + q \le 4$, the only (possibly) non-zero terms are $E^2_{n,0} \cong H_n(X)$ and $E^2_{0,4} = \Omega^{SO}_4$. Hence $E^2_{p,n-p} = E^\infty_{p,n-p}$ for n = 0, 1, 2, 3, and 4. From the spectral sequence one concludes

$$\Omega^{SO}_n(X) \cong H_n(X) \text{ for } n = 0,1,2,3$$

$$\Omega_n^{SO}(X) \cong Z \oplus H_4(X).$$

It can be shown that the map $\Omega_n^{SO}(X) \to H_n(X)$ is a Hurewicz map which takes $f: M \to X$ to $f_*([M])$. In particular this implies that any homology class in $H_n(X)$ for n = 0, 1, 2, 3, and 4 is represented by a map from an oriented manifold to X. The map $\Omega_4^{SO}(X) \to Z$ is the map taking $f: M \to X$ to the signature of M.

We next identify another edge homomorphism which can be constructed in the same manner as (9.5). The analysis will be slightly more involved and we will state it only in the case when G_* is ordinary homology with coefficients in an R-module (we suppress the coefficients).

In this context $E_{p,q}^2 = H_p(B; H_q F) = 0$ for $q < 0$ or $p < 0$. So $E_{*,*}^*$ is a first-quadrant spectral sequence, i.e. $E_{p,q}^r = E_{p,q}^\infty = 0$ for $q < 0$ or $p < 0$.

This implies that the filtration of $H_n(E)$ has finite length

$$0 = F_{-1,n+1} \subset F_{0,n} \subset F_{1,n-1} \subset \ldots\ldots \subset F_{n,0} = H_n(E)$$

Since

$$0 = E_{p,n-p}^\infty = F_{p,n-p} / F_{p-1,n-p+1}$$

if $p < 0$ or $n - p < 0$.

The second map in the short exact sequence

$$0 \to F_{n-1,1} \to F_{n,0} \to E_{n,0}^\infty \to 0$$

Can thus be thought of as a homomorphism

(9.6) $$H_n(E) \to E_{n,0}^\infty.$$

Lemma 9.11. There is an inclusion

$$E_{n,0}^\infty \subset E_{n,0}^2$$

for all n.

Proof. Since $E_{n+r,1-r}^r = 0$ for $r > 1$,

$$E_{n,0}^{r+1} = \frac{\ker d^r : E_{n,0}^r \to E_{n-r,r-1}^r}{\operatorname{Im} d^r : E_{n+r,1-r}^r \to E_{n,0}^r} = \ker d^r : E_{n,0}^r \to E_{n-r,r-1}^r.$$

Thus

$$\ldots.. \subset E_{n,0}^{r+1} \subset E_{n,0}^{r} \subset E_{n,0}^{r-1} \subset \ldots\ldots$$

And hence

$$E_{n,0}^{\infty} \bigcap_r E_{n,0}^{r} \subset E_{n,0}^{2}.$$

Note that the constant map from the fiber F to a point induces a homomorphism $H_n(B; H_0F) \to H_nB.$ If F is path connected, then the local coefficient system H_0F is trivial and $H_n(B; H_0(F)) = H_n(B)$ for all n.

Theorem 9.12. The composite map (also called an edge homomorphism)

$$H_n(E) = F_{n,0} \to E_{n,0}^{\infty} \subset E_{n,0}^{2} \cong H_n(B; H_0F) \to H_n(B)$$

is just the map induced on homology by the projection $f : E \to B$ of the fibration.

The long differential $d^k : E_{k,0}^k \to E_{0,k-1}^k$ in the spectral sequence for a fibration (for ordinary homology) has an alternate geometric interpretation called the transgression.

It is defined as follows. Suppose $f : E \to B$ is a fibration with fiber F. Fix $k > 0$. We assemble the homomorphism $f_* : H_k(E, F) \to H_k(B, b_0)$, the isomorphism $H_k(B) \cong H_k(B, b_0)$, and the connecting homomorphism $\delta : H_k(E, F) \to H_{k-1}(F)$ for the long exact sequence of the pair (E,F) to define a (not well-defined, multi–valued) function $\tau : H_k(B)" \to " H_{k-1}(F)$ as the “composite”

$$\tau : H_k(B) \cong H_k(B, b_0) \xrightarrow{f_*} H_k(E, F) \xrightarrow{\delta} H_{k-1}(F).$$

To make this more precise, we take as the domain of τ the image of f∗ : $f_* : H_k(E, F) \to H_k(B, b_0) \cong H_k(B)$, and as the range of τ the quotient of $H_{k-1}(F)$ by δ ($\ker f_* : H_k(E, F) \to H_k(B, b_0)$). A simple diagram chase shows τ is well-defined with this choice of domain and range.

Thus the transgression τ is an honest homomorphism from a subgroup of $H_k(B)$ to a quotient group of $H_{k-1}(F)$. intuitively, the transgression is trying his/her best to imitate the boundary map in the longexact homotopy sequence for a fibration (see Theorem 9.15 below).

Notes

Assume for simplicity that F is path connected, and consider the differential

$$d^k : E^k_{k,0} \to E^k_{0,k-1}$$

in the spectral sequence for this fibration (taking $G_* = H_* =$ ordinary homology). Its domain, $E^k_{k,0}$, is a subgroup of $E^2_{k,0} = H_k(B; H_0(F)) = H_k(B)$ because all differentials d^r into $E^k_{k,0}$ are zero for r < k (this is a first-quadrant spectral sequence) and hence $E^k_{k,0}$ is just the intersection of the kernels of $d^r : E^r_{k,0} \to E^r_{k-r,r-1}$ for $r < k$ Similarly the range $E^k_{0,k-1}$ of $d^k : E^k_{k,0} \to E^k_{0,k-1}$ is a quotient of $E^2_{0,k-1} = H_0(B; H_{k-1}(F))$, which by Proposition 5.14 is just the quotient of $H_{k-1}(F)$ by the action of $\pi_1(B)$.

We have shown that like the transgression, the differential $d^k : E^k_{k,0} \to E^k_{0,k-1}$ has domain identified with a subgroup of $H_k(B)$ and range a quotient of $H_{k-1}(F)$. The following theorem identifies the transgression and this differential.

Theorem 9.13 (Transgression Theorem). The differential $d^k : E^k_{k,0} \to E^k_{0,k-1}$ in the spectral sequence of the fibration $F \to E \to B$ coincides with the transgression

$H_k(B) \supset$ domain $(\tau) \xrightarrow{\tau}$ range $(\tau) = H_{k-1}(F)/\delta$ (ker f_*).

Check Your Progress

1. Prove: In the Leray-Serre-Atiyah-Hirzebruch spectral sequence there is a surjection

$$E^2_{0,n} \to E^\infty_{0,n} \quad \text{for all n.}$$

---

2. Explain about the edge homomorphism's and the transgression.

9.5 LET US SUM UP

1. A spectral sequence is a homological object of the following type:

One is given a sequence of chain complexes $\left(E^r, d^r\right)$ for r = 1, 2, . . .

2. A filtration of an R-module A is an increasing union

$$0 \subset \ldots\ldots \subset F_{-1} \subset F_0 \subset F_1 \subset \ldots\ldots \subset F_p \subset \ldots\ldots\ldots \subset A.$$

of submodules. A filtration is convergent if the union of the F_p's is A and their intersection is 0.

3. Given a filtration $F = \{F_n\}$ of an R-module A the associated graded module is the graded R-module denoted by Gr(A, F) and defined by

$$Gr(A,F)_p = \frac{F_p}{F_{p-1}}.$$

4. The differential $d^k : E^k_{k,0} \to E^k_{0,k-1}$ in the spectral sequence of the fibration $F \to E \to B$ coincides with the transgression

$H_k(B) \supset$ domain $(\tau) \xrightarrow{\tau}$ range $(\tau) = H_{k-1}(F)/\delta$ (ker f_*).

9.6 KEY WORDS

Spectral sequence

Leray-Serre-Atiyah-Hirzebruch spectral sequence

Edge homomorphism’s and the transgression

9.7 QUESTIONS FOR REVIEW

1. Explain about Spectral Sequence
2. Leray-Serre-Atiyah-Hirzebruch spectral sequence
3. Edge homomorphism’s and the transgression

9.8 SUGGESTIVE READINGS AND REFERENCES

1. Algebraic Topology – Satya Deo

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9.9 ANSWERS TO CHECK YOUR PROGRESS QUESTIONS

1. See section 9.4

2. See section 9.5

UNIT-10 APPLICATIONS OF THE HOMOLOGY SPECTRAL SEQUENCE

STRUCTURE

10.0 Objective

10.1 Introduction

10.2 The five term and serre exact sequences

10.3 Euler characteristics and fibrations

10.4 The homology gysin sequence

10.5 The cohomology spectral sequence

10.6 Homology of groups

10.7 Homology Of Covering Spaces

10.8 Relative spectral sequence

10.9 Let us sum up

10.10 Key words

10.11 Questions or review

10.12 Suggestive readings and references

10.13 Answers to check your progress

10.0 OBJECTIVE

In this unit we will learn and understand about the five-term and serre exact sequences, Enter characteristics and fibrations, The homology gysin sequence, The Cohomology special, Homology of groups, Homology of groups, Homology of covering spaces, Relative spectral sequences.

10.1 INTRODUCTION

In homological algebra and algebraic topology, a spectral sequence is a means of computing homology groups by taking successive approximations. Spectral sequences are a generalization of exact sequences, and since their introduction by Jean Leray (1946), they have become important computational tools, particularly in algebraic topology, algebraic geometry and homological algebra.

10.2 THE FIVE-TERM AND SERRE EXACT SEQUENCES

Corollary 10.1 : (Five-term exact sequence). Suppose that $F \to E \xrightarrow{f} B$ is a fibration with B and F path connected. Then there exists an exact sequence

$$H_2(E) \xrightarrow{f_*} H_2(B) \xrightarrow{\tau} H_0(B; H_1(F)) \to H_1(E) \xrightarrow{f_*} H_1(B) \to 0.$$

The composite of the surjection $H_1(F) \to H_0(B; H_1(F))$ with the map $H_0(B; H_1(F)) \to H_1(E)$ in this exact sequence is the homomorphism induced by the inclusion $F \to E$, and τ is the transgression.

Proof. Take $G_* = H_*(-)$, ordinary homology, perhaps with coefficients. The corresponding first quadrant spectral sequence has

$$E^2_{p,q} \cong H_p(B; H_q F)$$

and converges to $H_*(E)$.

The local coefficient system $\pi_1 B \to Aut(H_0(F))$ is trivial since F is path connected. Thus $E^2_{p,0} = H_p(B; H_0(F)) = H_p(B)$.

The following facts either follow immediately from the statement of previous Theorem or are easy to verify, using the bigrading of the differentials and the fact that the spectral sequence is a first–quadrant spectral sequence.

1. $H_1(B) \cong E^2_{1,0} = E^r_{1,0} = E^\infty_{1,0}$ for all $r \geq 2$.
2. $H_2(B) \cong E^2_{2,0}$.
3. $H_0(B; H_1 F) = E^2_{0,1}$.
4. $E^\infty_{2,0} = E^r_{2,0} = E^3_{2,0} = \ker d^2 : E^2_{2,0} \to E^2_{0,1}$ for all $r \geq 3$.
5. $E^\infty_{0,1} = E^r_{0,1} = E^3_{0,1} =$ coker $d^2 : E^2_{2,0} \to E^2_{0,1}$ for all $r \geq 3$..

Exercise : Prove these five facts.

The last two facts give an exact sequence

$$0 \to E^\infty_{2,0} \to E^2_{2,0} \xrightarrow{d_2} E^2_{0,1} \to E^\infty_{0,1} \to 0.$$

or, making the appropriate substitutions, the exact sequence

(10.2)

$$0 \to E^{\infty}_{2,0} \to H_2(B) \to H_0(B;H_1(F)) \to E^{\infty}_{0,1} \to 0.$$

Since the spectral sequence converges to $H_*(E)$, and the $E^{\infty}_{p,n-p}$ form the associated graded groups for $H_n(E)$, the two sequences

(10.3) $$0 \to E^{\infty}_{0,1} \to H_1(E) \to E^{\infty}_{1,0} \to 0$$

and

(10.4) $$E^{\infty}_{1,1} \to H_2(E) \to E^{\infty}_{2,0} \to 0$$

are exact.

Splicing the sequences (10.1), (10.2), and (10.3) together and using the first fact above one obtains the exact sequence

$$E^{\infty}_{1,1} \to H_2(E) \to H_2(B) \to H_0(B;H_1(F)) \to H_1(E) \to H_1(B) \to 0.$$

In this sequence the homomorphism $H_i(E) \to H_i(B)$ is the edge homomorphism and hence is induced by the fibration $f : E \to B$. The map

$H_0(B;H_1(F)) \to H_2(E)$ Composes with $H_1(F) \to H_0(B;H_1(F))$ to give the other edge homomorphism, induced by the inclusion of the fiber. The map

$H_2(B) \to H_0(B;H_1(F))$ is the transgression. These assertions follow by chasing definitions and using Theorems 10.5, 10.6 and 10.7.

We have seen, beginning with our study of the Puppe sequences, that cofibrations give exact sequences in homology and fibrations give exact sequences in homotopy. One might say that a map is a “fibration or cofibration in some range” if there are partial long exact sequences. Corollary 10.1 implies that if $\pi_1 B$ acts trivially on $H_1(F)$, then the fibration is a cofibration in a certain range. A more general result whose proof is essentially identical to that of Corollary 10.1 is given in the following important theorem.

Theorem 10.1 (Serre exact sequence). Let $F \xrightarrow{i} E \xrightarrow{f} B$ be a fibration

With B and F path connected and with $\pi_1 B$ acting trivially on $H_* F$. suppose $H_p B = 0$ for $0 < p < m$ and $H_q F = 0$ for $0 < q < n$. Then there is an exact sequence

$$H_{m+n-1}F \xrightarrow{i_*} H_{m+n-1}E \xrightarrow{f_*} H_{m+n-1}B \xrightarrow{\tau} H_{m+n-2}F \xrightarrow{i_*} \ldots\ldots.$$
$$\ldots.. \xrightarrow{f_*} H_1 B \to 0.$$

Exercise: Prove Theorem 10.1.

To understand this result, suppose B is (m − 1)-connected and F is $(n-1)$- connected. The long exact sequence for a fibration shows that E is (min(m, n) − 1)-connected, so that by the Hurewitz theorem, $H_q E = 0$ for q < min(m, n). So trivially the low-dimensional part of the Serre exact sequence is exact; indeed all groups are zero for q < min(m, n). The remarkable fact is that the sequence remains exact for all $\min\{m,n\} \le q < m+n$.

10.3 EULER CHARACTERISTICS AND FIBRATIONS

Let k be a field. Recall that the Euler characteristic of a space Z is defined to be the alternating sum $x(Z) = \sum_n (-1)^n \beta_n(Z;k)$ of the Betti numbers $\beta_n(Z;k) = \dim_k(H_n(Z;k))$ whenever this sum is a finite sum of finite ranks. For finite CW-complexes it is equal to the alternating sum of the number of n-cells by the following standard exercise applied to the cellular chain complex.

Exercise: Let (C_*, ∂) be a chain complex over a field with $\oplus_i C_i$ finite-dimensional. Show that the alternating sum of the ranks of the C_i equals the alternating sum of the ranks of the cohomology groups $H_i(C_*, \partial)$.

Given a product space $E = B \times F$ with B and F finite CW-complexes, the Kunneth theorem implies that the homology with field coefficients is a tensor product

$$H_*(E;k) \cong H_*(B;k) \otimes H_*(F;k)$$

from which it follows that the Euler characteristic is multiplicative

$$x(E) = x(B).\ x(F).$$

The following theorem extends this formula to the case when E is only a Product locally, i.e. fiber bundles, and even to fibrations.

Notice that the homology itself need not be multiplicative for a nontrivial fibration. For example, consider the Hopffibration $S^3 \to S^7 \to S^4$.

The graded groups $H_*\left(S^7;k\right)$ and $H_*\left(S^3;k\right) \otimes H_*\left(S^4;k\right)$ are not isomorphic, even though the Euler characteristics multiply (0 = 0 · 2).

Theorem 10.2 Let $p: E \to B$ be a fibration with fiber F, let k be a field, and suppose the action of $\pi_1 B$ on $H_*(F;k)$ is trivial. Assume that the Euler characteristics $x(B), x(F)$ are defined (e.g. if B,F are finite cell complexes). Then $x(E)$ is defined and

$$x(E) = x(B).\ x(F).$$

Proof. Since k is a field and the action of $\pi_1 B$ on $H_*(F;k)$ is trivial,

$$H_p\left(B;H_q(F;k)\right) \cong H_p(B;k) \otimes_k H_q(F;k)$$

by the universal coefficient theorem. Theorem 9.6 with $G_* = H_*(-;k)$ implies that there exists a spectral sequence with

$$E^2_{p,q} \cong H_p(B;k) \otimes H_q(F;k).$$

By hypothesis, $E^2_{p,q}$ is finite-dimensional over k, and is zero for all but finitely many pairs (p, q). This implies that the spectral sequence collapses at some stage and so $E^\infty_{p,q} = E^r_{p,q}$ for r large enough.

Define

$$E^r_n = \oplus_p E^r_{p,n-p}$$

For each n and $r \geq 2$ including $r = \infty$.

Then since the Euler characteristic of the tensor product of two graded vector spaces is the product of the Euler characteristics,

$$x\left(E^2_*\right) = x(B)\, x(F).$$

Notice that $\left(E^r_*, d^r\right)$ is a (singly) graded chain complex with homology E^{r+1}_*. Exercise: shows that for any $r \geq 2$,

$$x\left(E^r_*\right) = x\left(H_*\left(E^r_*, d^r\right)\right) = x\left(E^{r+1}_*\right).$$

Since the spectral sequence collapses $x\left(E^2_*\right) = x\left(E^\infty_*\right)$.

Since we are working over a field, $H_n(E;k)$ is isomorphic to its associated graded vector space $\oplus_p E^{\infty}_{p,n-p} = E^{\infty}_n$. In particular $H_n(E;k)$ is finite-dimensional and dim $H_n(E;k) = \dim E^{\infty}_n$.

Therefore,

$$x(B)x(F) = x(E^2_*) = x(E^{\infty}_*) = x(H_*(E;k)) = x(E).$$

10.4 THE HOMOLOGY GYSIN SEQUENCE

Theorem 10.3 Let R be a commutative ring. Suppose $f \to E \xrightarrow{f} B$ is a fibration, and suppose F is a R-homology n-sphere, i.e.

$$H_i(F;R) \cong \begin{cases} R & if\ i = 0\ or\ n, \\ 0 & otherwise. \end{cases}$$

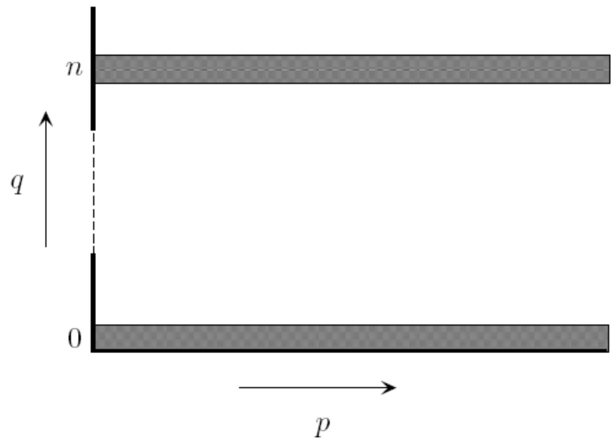
Assume that $\pi_1 B$ acts trivially on $H_n(F;R)$. Then there exists an exact sequence (R-coefficients):

$$\ldots. \to H_r E \xrightarrow{f_*} H_r B \to H_{r-n-1} B \to H_{r-1} E \xrightarrow{f_*} H_{r-1} B \to \ldots..$$

Proof. The spectral sequence for the fibration (using ordinary homology with R-coefficients) has

$$E^2_{p,q} \cong H_p(B; H_q F) = \begin{cases} H_p(B;R) & if\ \ q = 0\ \ or\ \ n, \\ \quad 0 & \quad otherwise. \end{cases}$$

The following diagram shows the E^2-stage. The two shaded rows (q = 0 and q = n) are the only rows that might contain a non-zero $E^2_{p,q}$.



Thus the only possibly non-zero differentials are

$$d^{n+1} : E^{n+1}_{p,0} \to E^{n+1}_{p-n-1,n}.$$

It follows that

$$E^{n+1}_{p,0} \cong E^2_{p,q} \cong H_p\left(B; H_q F\right) = \begin{cases} H_p B & \text{if } q = 0 \text{ or } n, \\ 0 & \text{otherwise.} \end{cases}$$

and

(10.5)

$$E^{\infty}_{p,q} \cong \begin{cases} 0 & \text{if } q \neq 0 \text{ or } n, \\ \ker\, d^{n+1} : E^{n+1}_{p,0} \to E^{n+1}_{p-n-1,n} & \text{if } q = 0 \\ \operatorname{coker}\, d^{n+1} : E^{n+1}_{p+n+1,0} & \text{if } q = n \end{cases}$$

The filtration of $H_r(E)$ reduces to

$$0 \subset E^{\infty}_{r-n,n} \cong F_{r-n,n} \subset F_{r,0} = H_r E$$

and so the sequences

$$0 \to E^{\infty}_{r-n,n} \to H_r E \to E^{\infty}_{r,0} \to 0$$

are exact for each r. Splicing these with the exact sequences

$$0 \to E^{\infty}_{p,0} \to E^{n+1}_{p,0} \xrightarrow{d^{n+1}} E^{n+1}_{p-n-1,n} \to E^{\infty}_{p-n-1,n} \to 0$$

(obtained from Equation (10.5)) gives the desired exact sequence

$$\ldots.. \to H_r E \xrightarrow{f_*} H_r B \to H_{r-n-1} B \to H_{r-1} E \to H_{r-1} B \to \ldots.$$

With the map labelled f_* induced by $f : E \to B$ by Theorem 9.12.

Exercise : Derive the Wang sequence. If $F \to E \to S^n$ is a fibration over S^n, then there is an exact sequence

$$\ldots.. \to H_r F \to H_r E \to H_{r-n} F \to H_{r-1} F \to \ldots\ldots$$

10.5 THE COHOMOLOGY SPECTRAL SEQUENCE

The examples in the previous section show that spectral sequences are a Useful tool for establishing relationships between the homology groups of the three spaces forming a fibration. Much better information can often be obtained by using the ring structure on cohomology. We next introduce the cohomology spectral sequence and relate the ring structures on cohomology and the spectral sequence. The ring structure makes the cohomology spectral sequence a much more powerful computational tool than the homology spectral sequence.

Definition: A bigraded spectral sequence $\left(E_r^{p,q}, d_r\right)$ is called a cohomology spectral sequence if the differential d_r has bidgree $(r, 1-r)$.

Notice the change in placement of the indices in the cohomology spectral sequence. The contravariance of cohomology makes it necessary to change the notion of a filtration. There is a formal way to do this, namely by "loweringindices", for example rewrite $H^p(X)$ as $H_{-p}(X)$, rewrite F^p as F_{-p}, Replace $E_r^{p,q}$ by $E_{-p,-q}^r$ and so forth. Unfortunately for this to work the notion of convergence of a spectral sequence has to be modified; with the definition we gave above the cohomology spectral sequence of a fibration will not converge. Rather than extending the formalism and making the notion of convergence technically more complicated, we will instead just make new definitions which apply in the cohomology setting.

Definition: A (cohomology) filtration of an R-module A is an increasing union

$$0 \subset \ldots\ldots \subset F^p \subset \ldots\ldots \subset F^2 \subset F^1 \subset F^0 \subset F^{-1} \subset \ldots. \subset A.$$

Of submodules. A filtration is convergent if the union of the F_p ' s is A and their intersection is 0.

If A itself is graded, then the filtration is assumed to preserve the grading i.e. $F^p \cap A^n \subset F^{p-1} \cap A^n$. If A is graded, then we bigrade the filtration by setting

$$F^{p,q} = F^q \cap A^{p+q}.$$

Definition: Given a cohomology filtration $F = \{F^n\}$ of an R-module A the associated graded module is the graded R-module denoted by Gr(A, F) and defined by

$$Gr(A, F)^p = \frac{F^p}{F^{p+1}}.$$

Definition: Given a bigraded cohomology spectral sequence $\left(E_r^{p,q}, d_r\right)$, and a graded R-module A^*, we say the spectral sequence converges to A^* and write

$$E_2^{p,q} \Rightarrow A^{p+q}$$

if:

1. for each (p, q) there exists an r_0 so that $d_r : E_r^{p-r,q+r-1} \to E_r^{p,q}$ is zero for all $r \geq r_0$; in particular there is an injection $E_{r+1}^{p,q} \to E_r^{p,q}$ for all $r \geq r_0$, and

2. there is a convergent filtration of A^*, so that for each n, the limit $E_\infty^{p,q} = \cap_{r \geq r_0} E_r^{p,q}$ is isomorphic to the associated graded $Gr\left(A^*\right)^p$.

Theorem 10.4. Let $F \to E \xrightarrow{f} B$ be a fibration, with B a path connected CW-complex. Let G^* be an additive cohomology theory. Assume either that B is a finite-dimensional CW-complex or else that there exists an N so that $G^q\left(F\right) = 0$ for all $q < N$. Notice that $\pi_1\left(B\right)$ acts on $G^q\left(F\right)$ determining a local coefficient system.

Then there exists a (cohomology) spectral sequence

$$H_p\left(B; G^q F\right) \cong E_2^{p,q} \Rightarrow G^{p+q}\left(E\right).$$

There is a version of this theorem which applies to infinite CW-complexes.

Exercise:. State and prove the cohomology versions of the Serre, Gysin, and Wang sequences. Construct the cohomology edge homomorphisms and the cohomology transgression and state the analogues of Theorems 10.1, 10.2, and 10.3.

As an example we show how to compute the complex K-theory of complex projective space CP^k

The computation of complex K-theory was the original motivation for Atiyah-Hirzebruch to set up their spectral sequence. Complex K-theory is a cohomology theory satisfying $K^n\left(X\right) = K^{n+2}\left(X\right)$, and its coefficients are given by

$$K^{2n}\left(pt\right) = \pi_0\left(Z \times BU\right) = Z$$

and

$$K^{2n+1}\left(pt\right) = \pi_1\left(Z \times BU\right) = 0.$$

Theorem 10.4, applied to the trivial fibration

$$pt \to CP^k \xrightarrow{Id} CP^k,$$

says there exists a cohomology spectral sequence $E_r^{p,q}$ satisfying

$$H^p\left(CP^k;K^q\left(pt\right)\right)\cong E_2^{p,q}\Rightarrow K^{p+q}\left(CP^k\right).$$

The coefficients are untwisted since the fibration is trivial. Since

$$H^p\left(CP^k\right)=\begin{cases} Z & if\ p\, is\, even, 0\le p\le 2k \\ 0 & otherwise. \end{cases}$$

it follows that

$$E_2^{p,q}=\begin{cases} Z & if\ p\, and\ q\, are\ even\, 0\le p\le 2k \\ 0 & otherwise. \end{cases}$$

This checkerboard pattern forces every differential to be zero, since one of the integers $(r,1-r)$ must be odd! Notice, by the way, that this is not a first-quadrant spectral sequence since the K-theory of a point is non-zero in positive and negative dimensions.

Therefore $E_2^{p,q}=E_\infty^{p,q}$ and the associated graded group to $K^n\left(CP^k\right), \oplus_p E_\infty^{p,n-p}$, is a direct sum of $k+1$ copies of Z, one for each pair (p,q) so that $p+q=n$, both p and q are even, and $0\le p\le 2k$. Inducting down the filtration we see that $K^n\left(CP^k\right)$ has no torsion and hence is isomorphic to its associated graded group. Therefore

$$K^n\left(CP^k\right)=\begin{cases} Z^{k+1} & if\ n\, is\, even, \\ 0 & otherwise. \end{cases}$$

To study the multiplicative properties of the cohomology spectral sequence, take G^* to be ordinary cohomology with coefficients in a commutative ring $G^*(E)=G^*(E;R)$. Let $F\to E\to B$ be a fibration. To avoid working with cup products with local coefficients, we assume that $\pi_1 B$ acts trivially on $H^*(F)$.

Lemma 10.5. $H^p\left(B;H^qF\right)\cong E_2^{p,q}$ is a bigraded algebra over R.

Proof. The cup product on H^*B induces a bilinear map

$$H^p\left(B;H^qF\right)\times H^r\left(B;H^8F\right)\to H^{p+r}\left(B;H^qF\otimes H^8F\right).$$

Composingwith the coefficient homomorphism induced by the cup product on $H*(F)$

$$H^q(F)\otimes H^8(F)\to H^{q+8}(F)$$

gives the desired multiplication

$$E_2^{p,q} \otimes E_2^{r,8} = H^p\left(B; H^q F\right) \otimes H^r\left(B; H^8 F\right) \to H^{p+r}\left(B; H^{q+8} F\right) = E_2^{p+r,q+8}.$$

In many contexts the map $E_2^{*,0} \otimes E_2^{0,*} \to E_2^{*,*}$ is an isomorphism. Theorem 2.33 can be quite useful in this regard, For example if R is a field and B and F are simply connected finite CW-complexes then the map is an isomorphism.

Theorem 10.6 The (Leray-Serre) cohomology spectral sequence of the fibration is a spectral sequence of R-algebras. More precisely:

1. $E_t^{*,*}$ is a bigraded R-algebra, i.e. there are products

$$E_t^{p,q} \times E_t^{r,s} \to E_t^{p+r,q+s}.$$

2. $d_t : E_t \to E_t$ is a derivation. This means that if $a \in E_t^{p,q}, b \in E_t^{r,s}$

$$d_t\left(a.b\right) = \left(d_t a\right).b + \left(-1\right)^{p+q} a.d_t b.$$

3. The product on E_{t+1} is induced from the one on E_t starting with the product on E_2 given by cup products, as in Lemma

4. The following two ring structures on E_∞ coincide. (This assertion is a compatibility condition which relates the cup products on B, F, and E.)

(a) Make $E_\infty^{*,*}$ a bigraded R-algebra by using that each $\left(a,b\right) \in E_\infty^{p,q} \times E_\infty^{r,s}$ is represented by an element of $E_t^{p,q} \times E_t^{r,s}$ for t large enough.

(b) The (usual) cup product

$$\cup : H^*\left(E\right) \times H^*\left(E\right) \to H^*\left(E\right)$$

is "filtration preserving", i.e. the diagram

$$\begin{tikzcd}
F^{p,q} \times F^{r,s} \arrow[r] \arrow[d] & F^{p+r,q+s} \arrow[d] \\
H^{p+q}E \times H^{r+s}E \arrow[r, "\cup"] & H^{p+q+r+s}E
\end{tikzcd}$$

commutes (this comes from the construction of the filtration), and so this cup product induces a product on the associated graded module, i.e. on E_∞. .

Notes

Exercise : Suppose that E is a graded ring and $d:E\to E$ is a differential $\left(d^2=0\right)$ and a derivation (Equation 9.11), then show that the cohomology $H^*(E,d)$ inherits a graded ring structure.

Exercise : Show that a filtration-preservingm ultiplication on a filtered algebra induces a multiplication on the associated graded algebra.

Proposition : The rational cohomology ring of K(Z, n) is a polynomial ring on one generator if n is even and a truncated polynomial ring one one generator (in fact an exterior algebra on one generator) if n is odd:

$$H^*\left(K(Z,n);Q\right)=\begin{cases} Q[l_n] & \text{if } n \text{ is even,} \\ Q[l_n]/l_n^2 & \text{if } n \text{ is odd} \end{cases}$$

Where $\deg(l_n)=n.$

Proof. We induct on n. For $n=1, k(Z,1)=S^1$ which has cohomology ring $Z[l_1]/l_1^2$.

Suppose the theorem is true for $k<n$. Consider the Leray–Serre spectral sequence for path space fibration $K(Z,n-1)\to P\to K(Z,n)$ for cohomology with rational coefficients. Then

$$E_2^{p,q}=H^p\left(K(Z,n);Q\right)\otimes_Q H^q\left(K(Z,n-1);Q\right)\Rightarrow H^{p+q}(P;Q).$$

Since $H^{p+q}(P,Q)=0$ for $(p,q)\neq(0,0)$, The differential

$$d_n:E_n^{0,n-1}\to E_n^{n,0}$$

must be an isomorphism. Since $E_n^{0,n-1}=H^{n-1}\left(K(Z,n-1);Q\right)\cong Q,$ generated by l_{n-1}, and $E_n^{n,0}=E_2^{n,0}=H^n\left(K(Z,n);Q\right)\cong Q,$ generated by l_n, it follows that $d_n(l_{n-1})$ is a non-zero multiple of l_n. By rescaling the generator l_n by a rational number assume inductively that $d_n(l_{n-1})=l_n.$

Consider the cases n even and n odd separately. If n is even, then since $H^q\left(K(Z,n-1);Q\right)=0$ unless $q=0$ or $n-1, E_2^{p,q}=0$ unless $q=0$ or $n-1$. This implies that $0=E_\infty^{p,q}=E_{n+1}^{p,q}$ for $(p,q)\neq(0,0)$ and the derivation property of d_n says that $d_n\left(l_{n-1}l_n^r\right)=l_n^{r+1}$ which, by induction on r, is non-zero. It follows easily from $0=E_\infty^{p,q}=E_{n+1}^{p,q}$ for

$(p,q)\neq(0,0)$ that $H^p(K(Z,n);Q)=0$ if p is not a multiple of n, and is isomorphic to Q for $p=nr$. Since l_n^r is non-zero it generates $H^{nr}(K(Z,n);Q)\cong Q$ and so $H^*(K(Z,n);Q)$ is a polynomial ring on l_n as required.

If n is odd, the derivation property of d_n implies that

$$d_n(l_{n-1}^2)=d_n(l_{n-1})l_{n-1}+(-1)^{n-1}l_{n-1}d_n(l_{n-1})=2l_{n-1}l_n.$$

Hence $d_n:E_n^{0,2n-2}\to E_2^{n,n-1}$ is an isomorphism. More generally by induction one sees that $d_n(l_{n-1}^r)=rl_nl_{n-1}^{r-1}$, so that $d_n:E_n^{0,r(n-1)}\to E_2^{n,(r-1)(n-1)}$ is an isomorphism. It is then easy to see that the spectral sequence collapses at E_{n+1}, and hence $H^p(K(Z,n);Q)=Q$ for p = 0 or n and zero otherwise.

We will show how to use Theorem 154 to compute π_4S^3. This famous Theorem was first proven by G.W. Whitehead and Rohlin (independently). The argument is effortless using spectral sequences.

Theorem 10.6 $\pi_4S^3=Z/2$.

Proof. Since $Z=H^3(S^3)=[S^3,k(Z,3)]$, choose a map $f:S^3\to K(Z,3)$ representing the generator. For example, $K(Z,3)$ can be obtained by adding 5 cells, 6 cells, etc. to S^3 inductively to kill all the higher homotopy groups of S^3 and then f can be taken to be the inclusion. The Hurewicz theorem implies that $f_*:\pi_3S^3\to\pi_3(K(Z,3))$ is an isomorphism.

Pull back the fibration

$$K(Z,2)\to *\to K(Z,3)$$

(this is shorthand for $\Omega K(Z,3)\to P\to K(Z,3)$ where P is the contractible path space) via f to get a fibration

$$K(Z,2)\to X\to S^3.$$

Alternatively, let X be the homotopy fiber of f, i.e. $X\to S^3\to K(Z,3)$ is a fibration up to homotopy. Then $\Omega K(Z,3)\simeq K(Z,2)$ is the homotopy fiber of $X\to S^3$

In the longexact homotopy sequence for a fibration, $\partial : \pi_3 S^3 \xrightarrow{\cong} \pi_2(K(Z,2))$. Hence

$$\pi_k X = \begin{cases} 0 & \text{if } k \le 3, \\ \pi_k S^3 & \text{if } k > 3. \end{cases}$$

In particular, $H_4 X = \pi_4 X = \pi_4 S^3$. We will try to compute $H_4 X$ using a spectral sequence.

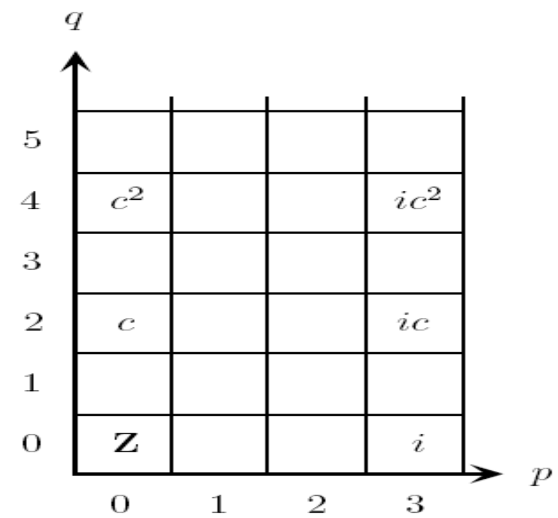
Consider the cohomology spectral sequence for the fibration (9.12). Then $E_2^{p,q} = H^p(S^3; H^q K(Z,2))$. Recall that $K(Z,2)$ is the infinite complex projective space CP^∞ whose cohomology algebra is the 1-variable polynomial ring $H^*(K(Z,2)) = Z[c]$ where $\deg(c) = 2$.

Exercise: Give another proof of the fact that $H^*(K(Z,2)) = Z[c]$ using the spectral sequence for the path space fibration

$$K(Z,1) \to * \to K(Z,2)$$

and the identification of $K(Z,1)$ with S^1.

Let $i \in H^3(S^3)$ denote the generator. Then the E_2 -stage in the spectral sequence is indicated in the following diagram. The labels mean that the groups in question are infinite cyclic with the indicated generators. The empty entries are zero. The entries in this table are computed using Lemma 9.23.



Since $H^2 X = 0 = H^3 X$ it follows that $d^3 c = i$. Therefore,

$$d^3 c^2 = ic + ci = 2ci$$

This implies that $Z/2 \cong E_4^{3,2} = E_\infty^{3,2} \cong H^5 X$ and $0 = E_4^{0,4} = E_\infty^{0,4} = H^4 X.$

The universal coefficient theorem implies that $H_4 X = Z/2.$ We conclude that $Z/2 \cong \pi_4 X = \pi_4 S^3,$ as desired.

Corollary 10.9 $\pi_{n+1} S^n = Z/2$ for all $n \geq 3.$ In particular, $\pi_1^S = Z/2.$

Proof. This is an immediate consequence of the Freudenthal suspension theorem (Theorem 8.7).

Corollary 10.10 $\pi_4 S^2 = Z/2.$

Proof. Apply the longexact sequence of homotopy groups to the Hopf fibration $S^1 \to S^3 \to S^2.$

The reader should think about the strategy used to make these computations. On the one hand fibrations were used to relate homotopy groups of various spaces; on the other spectral sequences are used to compute homology groups. The Hurewitz theorem is then used to conclude that a homology group computation in fact gives a homotopy group computation.

10.6. HOMOLOGY OF GROUPS

Definition : Let G be a group. Define the cohomology of G with Z coefficients by

$$H^k(G;Z) = H^k(K(G,1);Z).$$

Similarly define the homology of G

$$H^k(G;Z) = H_k(K(G,1);Z).$$

More generally define the homology and cohomology of G with coefficients in any R-module A to be the corresponding homology or cohomology of $K(G,1)$.

Corollary 10.27 implies that the homology and cohomology of a group are well-defined. Moreover, the assignment $G \to K(G,1)$ is functorial and takes short exact sequences to fibrations. (The functoriality can be interpreted in two different ways. For every group one associates a homotopy type of spaces, and a group homomorphism leads to a homotopy class of maps between the spaces. Alternatively, one can construct an honest functor from the category of groups to the category of spaces by giving a specific model of K(G, 1) related to the bar resolution in homological algebra.)

Groups are very mysterious nonabelian things and thus are hard to study. Homology of groups gives abelian invariants, and has been very useful in group theory as well as topology.

It follows that to understand the homology of groups related by exact sequences amounts to understanding the homology of a fibration, for which, as we have seen, spectral sequences are a good tool.

It is easy to see that $K(A \times B, 1) = K(A,1) \times K(B,1)$, and so the Kunneth theorem can be used to compute the cohomology of products of groups. Therefore the following result is all that is needed to obtain a complete computation of the cohomology of finitely generated abelian groups.

Theorem 10.9 The cohomology of Z/n is given by

$$H^q(Z/n; Z) = \begin{cases} Z & \text{if } q = 0, \\ 0 & \text{if } q \text{ is odd, and} \\ Z/n & \text{if } q > 0 \text{ is even} \end{cases}$$

Proof. The exact sequence $0 \to Z \xrightarrow{\times n} Z \to Z/n \to 0$ induces a fibration sequence

$$K(Z,2) \to K(Z,2) \to K(Z/n,2)$$

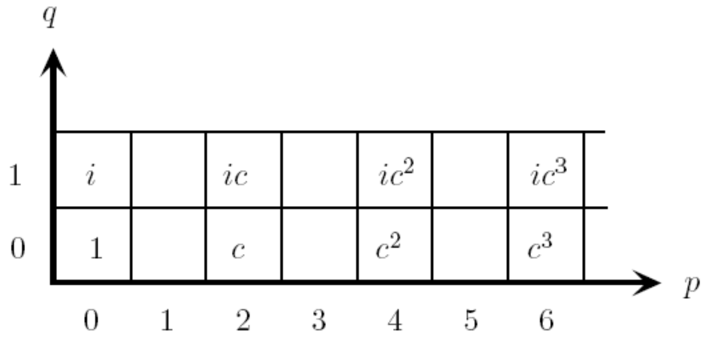
(see Proposition 10.28). By looping this fibration twice (i.e. taking iterated homotopy fibers twice; see Theorem 6.40) we obtain the fibration

$$K(Z,1) \to K(Z/n,1) \to K(Z,2).$$

The fiber $K(Z,1)$ is a circle. Consider the spectral sequence for this fibration. The base is simply connected so there is no twisting in the coefficients. Notice that

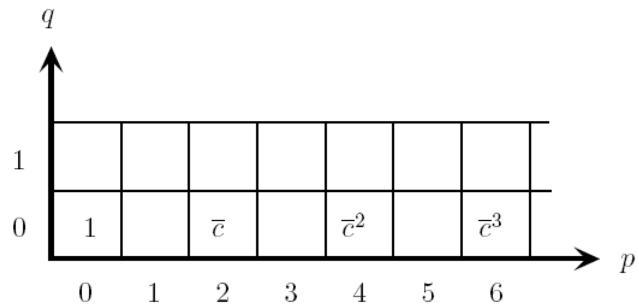
$$E_2^{p,q} = H^p\left(K(Z,2); H^q S^1\right) = \begin{cases} 0 & \text{if } q > 1, \text{and} \\ H^p\left(K(Z,2); Z\right) & \text{if } q = 0 \text{ or } 1. \end{cases}$$

Using Lemma 10.23, the E_2 -stage is given by the following table, with the empty entries equal to 0 and the others infinite cyclic with the indicated generators (where i is the generator of $H^1\left(S^1\right)$).



Of course $d^2(i) = kc$ for some integer k, and the question is: what might k be? We can find out by "peeking at the answer." Since $E_\infty^{0,2} = 0 = E_\infty^{1,1}$, we see that $H^2\left(K\left(Z/n,1\right)\right) = E_\infty^{2,0} \cong Z/k$. Since $\pi_1\left(K\left(Z/n,1\right)\right) = Z/n$, by the universal coefficient theorem, we see that H^2 must be Z/n and hence $k = \pm n$. (Neat, huh?)

Let $\bar{c}$ be the image of c in $E_3^{2,0}$. Here is a picture of the E^3 -stage.



From this we see that the spectral sequence collapses at E^3, and that as graded rings $E_\infty^{*,0} \cong H^*\left(K\left(Z/n,1\right)\right)$. This not only completes the proof of the theorem, but also computes the cohomology ring

$$H^*\left(K\left(Z/n,1\right)\right) = Z\left[\bar{c}\right]/\left\langle n\bar{c}\right\rangle.$$

Also, we can get the homology from the cohomology by using the universal coefficient theorem:

$$H_q\left(Z/n\right) = \begin{cases} Z & \text{if } q = 0, \\ Z/n & \text{if } q \text{ is odd, and} \\ 0 & \text{if } q > 0 \text{ is even.} \end{cases}$$

In applications, it is important to know the mod p-cohomology ring (which is the mod p-cohomology ring on an infinite-dimensional lens space). By the K¨unneth theorem (which implies that, with field coefficients, $H^*(X \times Y) \cong H^*(X) \otimes H^*(Y)$), it suffices to consider the case where n is a prime power. Let F_p denote the field Z/pZ for a prime p.

Exercise: Show that $H^*(Z/2;F_2) \cong F_2[a]$ where a has degree one, and if $p^k \neq 2, H^*(Z/p^k;F_p) \cong \Lambda(a) \otimes F_p[b]$, where a has degree one and b has degree 2. Here $\Lambda(a)$ is the 2-dimensional graded algebra over F_p with $\Lambda(a)^1 \cong F_p$ with generator 1, and $\Lambda(a)^1 \cong F_p$ with generator a. (Hint: Use $RP^\infty = K(Z/2,1)$ and that $a.a = -a.a$ for $a \in H^1$.)

Exercise : Compute $H^p(K(Z/2,n);Z/2)$ for as many p and n as you can. Hint: try induction on n, using the fibration

$$K(Z/2,n) \to * \to K(Z/2,n+1).$$

10.7 HOMOLOGY OF COVERING SPACES

Suppose that $f : X \to X$ is a regular cover of a path connected space X. Letting $G = \pi_1(X)/f_*(\pi_1(X)), f : X \to X$ is a principal G-bundle (with G discrete). Thus $G \to X \to X$ is pulled back from the universal G-bundle

$G \to EG \to BG$. In other words, there is a diagram

$$\begin{tikzcd}
G \arrow[r, "="] \arrow[d] & G \arrow[d] \\
\tilde{X} \arrow[r] \arrow[d, "f"'] & EG \arrow[d] \\
X \arrow[r, "h"] & BG.
\end{tikzcd}$$

It follows that the sequence

$$X \to X \to BG$$

is a fibration (up to homotopy). (One way to see this is to consider the Borel fibration $X \to X \times_G EG \to X$. Since G acts freely on X, there is another fibration $EG \to X \times_G EG \to X/G$. Since EG is contractible we see that the total space of the Borel fibration is homotopy equivalent to X.) Since G is discrete, $BG = K(G,1)$. Applying the homology (or cohomology) spectral sequence to this fibration immediately gives the following spectral sequence of a covering space (we use the notation $H_*(G) = H_*(K(G,1))$).

Theorem 10.10 given a regular cover $f : X \to X$ with group of covering automorphisms $G = \pi_1(X)/f_*(\pi_1(X))$, there is a homology spectral sequence

$$H_p\left(G; H_q\left(X\right)\right) \cong E^2_{p,q} \Rightarrow H_{p+q}(X)$$

and a cohomology spectral sequence

$$H^P\left(G; H^q\left(X\right)\right) \cong E_2^{p,q} \Rightarrow H^{p+q}(X).$$

The twisting of the coefficients is just the one induced by the action of G on X by covering transformations.

Applying the five-term exact sequence (Corollary 9.14) in this context gives the very useful exact sequence

$$H_2(X) \to H_2(G) \to H_0\left(G; H_1\left(X\right)\right) \to H_1(X) \to H_1(G) \to 0.$$

Exercise : Use the spectral sequence of the universal cover to show that for a path connected space X the sequence

$$\pi_2(X) \xrightarrow{\rho} H_2(X) \to H_2\left(\pi_1(X)\right) \to 0$$

is exact, where ρ denotes the Hurewicz map.

As an application we examine the problem of determiningwhic h finite groups G can act freely on S^k. Equivalently, what are the fundamental groups of manifolds covered by the k-sphere? First note that if $g : S^k \to S^k$ is a fixed-point free map then g is homotopic to the antipodal map (can you

remember how to prove this?), and so is orientation-preserving if k is odd and orientation-reversing if k is even. Thus if k is even, the composite of any two non-trivial elements of G must be trivial, from which it follows that

G has 1 or 2 elements. We shall henceforth assume k is odd, and hence that

G acts by orientation-preserving fixed-point free homeomorphisms.

Thus the cohomology spectral sequence for the cover has

$$E_2^{p,q} = \begin{cases} H^p\left(G; H^q\left(S^k\right)\right) = H^p(G) & if \ \ q = 0 \, or \, q = k \\ 0 & otherwise \end{cases}$$

and converges to $H^{p+q}\left(S^k / G\right)$. This implies that the only possible non-zero differentials are

$$d_k : E_{k+1}^{p,k} \to E_{k+1}^{p-k-1,0}$$

and that the spectral sequence collapses at E_{k+2}.

Notice that S^k / G is a compact manifold of dimension k, and in particular $H^n\left(S^k / G\right) = 0$ for $n > k$. This forces $E_\infty^{p,q} = 0$ whenever $p + q > k$. Hence the differentials $d_k : E_{k+1}^{p,k} \to E_{k+1}^{p+k+1,0}$ are isomorphisms for $p \geq 1$, and since these are the only possible non-zero differentials we have

$$E_{k+1}^{p,k} = E_2^{p,k} \cong H^p(G) \text{ and } E_{k+1}^{p+k+1,0} = E_2^{p+k+1,0} \cong H^{p+k+1}(G)$$

so that $H^p(G) \cong H^{p+k+1}(G)$ for $p \geq 1$.

Thus G has periodic cohomology with period $k+1$. Any subgroup of G also acts freely on S^k by restrictingthe action. This implies the following theorem.

Theorem 10.11 If the finite group G acts freely on an odd-dimensional sphere S^k, then every subgroup of G has periodic cohomology of period $k+1$.

As an application, first note the group $Z/p \times Z/p$ does not have periodic cohomology; this can be checked using the Kunneth theorem. We conclude that any finite group acting freely on a sphere cannot contain a subgroup isomorphic to $Z/p \times Z/p$.

10.8 RELATIVE SPECTRAL SEQUENCES

In studying maps of fibrations, it is useful to have relative versions of the homology and cohomology spectral sequence theorems. There are two relative versions, one involving a subspace of the base and one involving a subspace of the fiber.

Theorem 10.12 Let $F \to E \xrightarrow{f} B$ be a fibration with B a CW-complex.

Let $A \subset B$ a subcomplex. Let $D = p^{-1}(A)$

1. There is a homology spectral sequence with

$$H_p\left(B, A; G_q F\right) \cong E^2_{p,q} \Rightarrow G_{p+q}(E, D).$$

2. If B is finite-dimensional or if there exist an N so that $G^q(F) = 0$ for all $q < N$, there is a cohomology spectral sequence with

$$H^p\left(B, A; G^q F\right) \cong E_2^{p,q} \Rightarrow G^{p+q}(E, D).$$

Theorem 10.13. Let $F \to E \xrightarrow{f} B$ be a fibration with B a CW-complex.

Let $E_0 \subset E$ so that $f|_{E_0} : E_0 \to B$ is a fibration with fiber F_0.

1. There is a homology spectral sequence with

$$H_p\left(B; G_q(F, F_0)\right) \cong E^2_{p,q} \Rightarrow G_{p+q}(E, E_0).$$

2. If B is finite-dimensional or if there exist an N so that $G^q(F, F_0) = 0$ for all $q < N$, there is a cohomology spectral sequence with

$$H^p\left(B; G^q(F, F_0)\right) \cong E_2^{p,q} \Rightarrow G^{p+q}(E, E_0).$$

Check Your Progress

1. Prove $H^p\left(B; H^q F\right) \cong E_2^{p,q}$ is a bigraded algebra over R

2. Prove The rational cohomology ring of K(Z, n) is a polynomial ring on one generator if n is even and a truncated polynomial ring one one generator (in fact an exterior algebra on one generator) if n is odd:

$$H^*\left(K(Z,n);Q\right)=\begin{cases} Q[l_n] & if\ n\ is\ even, \\ Q[l_n]/l_n^2 & if\ n\ is\ odd \end{cases}$$

Where $\deg(l_n)=n.$

3. Explain about homology of covering spaces

10.9 LET US SUM UP

1. Let $p:E\to B$ be a fibration with fiber F, let k be a field, and suppose the action of $\pi_1 B$ on $H_*(F;k)$ is trivial. Assume that the Euler characteristics $x(B), x(F)$ are defined (e.g. if B,F are finite cell complexes). Then $x(E)$ is defined and

$x(E)=x(B).\ x(F).$

2. A bigraded spectral sequence $\left(E_r^{p,q}, d_r\right)$ is called a cohomology spectral sequence if the differential d_r has bidgree $(r,1-r)$.

3. Given a cohomology filtration $F=\{F^n\}$ of an R-module A the associated graded module is the graded R-module denoted by Gr(A, F) and defined by

$$Gr(A,F)^p=\frac{F^p}{F^{p+1}}.$$

10.10 KEY WORDS

Homology Gysin sequence

Cohomology spectral sequence

Homology of groups

Homology of covering sequences

10.11 QUESTIONS FOR REVIEW

1. Explain about Euler characteristics and fimbriations.
2. Explain about the Cohomology spectral sequence
3. Explain about homology of covering spaces

10.12 SUGGESTIVE READINGS AND REFERENCES

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10.13 ANSWERS TO CHECK YOUR PROGRESS

1. See section 10.6
2. See section 10.6
3. See section 10.8

UNIT- 11. COMPLEX MAINFOLDS AND VECTOR BUNDLES

STRUCTURE

11.0 Objective

11.1 Introduction

11.2 Holomorphic functions

11.3 Complex manifolds

11.4 Sub manifolds

11.5 Properties of complex manifolds

11.5 Dolbeault Cohomology

11.7 Holomorphic vector bundles

11.8 Chern classes of line bundles

11.9 Chern classes of vector bundles

11.10 Kodaira- Serre duality

11.11 Connections

11.12 Let us sum up

11.13 Key words

11.14 Questions for review

11.15 Suggestive readings and references

11.16 Answers to check your progress

11.0 OBJECTIVE

In this unit we will learn and understand about Holomorphic functions, Complex manifolds, Properties of complex manifolds, Dolebeault Cohomology, Holomorphic vector bundles and Connections.

11.1 INTRODUCTION

In this unit we give a sketchy introduction to complex manifolds. The reader is assumed to be acquainted with the rudiments of the theory of differentiable manifolds.

11.2 HOLOMORPHIC FUNCTIONS

Let $U \subset C$ be an open subset. We say that a function $f : U \to C$ is holomorphic if it is C^1 and for all $x \in U$ its differential $Df_x : C \to C$ is not only R-linear but also C-linear. If elements in C are written $z = x + iy$, and we set $f(x, y) = \alpha(x, y) + i\beta(x, y)$, then this condition can be written as

(11.1) $$\alpha_x = \beta_y, \qquad \alpha_y = \beta_x$$

(these are the Cauchy-Riemann conditions). If we use $z, \bar{z}$ as variables, the Cauchy-Riemann conditions read $f_{\bar{z}} = 0$, i.e. the holomorphic functions are the C^1 function of the variable z. Moreover, one can show that holomorphic functions are analytic.

The same definition can be given for holomorphic functions of several variables.

Definition : Two open subsets U, V of C^n are said to biholomorphic if there exists a bijective holomorphic map $f : U \to V$ whose inverse is holomorphic. The map f itself is then said to be biholomorphic.

11.3 COMPLEX MANIFOLDS

Complex manifolds are defined as differentiable manifolds, but requiring that the local model is C^n, and that the transition functions are

$$\psi_i \circ \psi_j^{-1} : \psi_i\left(U_i \cap U_j\right) \to \psi_i\left(U_i \cap U_j\right)$$

biholomorphic.

Definition 11.2. An n-dimensional complex manifold is a second countable Hausdor topological space X together with an open cover $\{U_i\}$ and maps $\psi_i : U_i \to C^n$ which are homeomorphisms onto their images, and are such that all transition functions are biholomorphisms.

Example 11.3. (The Riemann sphere) Consider the sphere in R^3 centered at the origin and having radius $\frac{1}{2}$, and identify the tangent planes at

$\left(0,0,\frac{1}{2}\right)$ and $\left(0,0,-\frac{1}{2}\right)$ with C. The stereographic projections give local complex coordinates z_1, z_2; the transition function $z_2 = 1/z_1$ is defined in $C^* = C - \{0\}$ and is bi holomorphic.

1-dimensional complex manifolds are called Riemann surfaces. Compact Riemann surfaces play a distinguished role in algebraic geometry; they are all algebraic (i.e. they are sets of zeroes of systems of homogeneous polynomials), as we shall see in Chapter 8.

Example 11.4. (Projective spaces) We define the n-dimensional complex projective space as the space of complex lines through the origin of C^{n+1}, i.e.

$$P_n = \frac{C^{n+1} - \{0\}}{C^*}.$$

By standard topological arguments P_n with the quotient topology is a Hausdorff second countable space.

Let $\pi : C^{n+1} - \{0\} \to P_n$ be the projection, If $w = \left(w^0, \ldots\ldots, w^n\right) \in C^{n+1}$ we shall denote $\pi(w) = \left[w^0, \ldots\ldots, w^n\right]$. The numbers $\left(w^0, \ldots\ldots, w^n\right)$ are said to be the homogeneous coordinates of the point $\pi(w)$. If $\left(u^0, \ldots\ldots, u^n\right)$ is another set of homogeneous coordinates for $\pi(w)$, then $u^i = \lambda w^i$, with $\lambda \in C^* \left(i = 0, \ldots\ldots, n\right)$.

Denote by $U_i \subset C^{n+1}$ the open set where $w^i \neq 0$, let $U_i = \pi\left(U_i\right)$, and define a map

$$\psi_i : U_i \to C^n, \qquad \psi\left(\left[w^0, \ldots\ldots., w^n\right]\right) = \left(\frac{w^0}{w^i}, \ldots.., \frac{w^{i-1}}{w^i}, \frac{w^{i+1}}{w^i}, \ldots\ldots., \frac{w^n}{w^i}\right).$$

The sets U_i cover P_n, the maps ψ_i are homeomorphisms, and their transition functions

$$\psi_i \circ \psi_j^{-1} : \qquad \psi_j\left(U_j\right) - \psi_i\left(U_i\right),$$

$$\psi_i \circ \psi_j^{-1}\left(z^1, \ldots\ldots, z^n\right) = \left(\frac{z^1}{z^i}, \ldots\ldots, \frac{z^{i-1}}{z^i}, \frac{z^{i+1}}{z^i}, \ldots\ldots \frac{1}{z^i}, \ldots\ldots \frac{z^n}{z^i}\right),$$

$\uparrow$

j-th argument

are biholomorphic, so that P_n is a complex manifold (we have assumed that $i < j$). The map π restricted to the unit sphere in C^{n+1} is surjective, so that P_n is compact. The previous formula for n = 1 shows that P_n is biholomorphic to the Riemann sphere.

The coordinates defined by the maps ψ_i, usually denoted $\left(z^1, \ldots\ldots, z^n\right)$, are called affine or Euclidean coordinates.

Example 11.5. (The general linear complex group). Let

$M_{k,n}$ = { $k \times n$ matrices with complex entries, $k \le n$ }

$M_{k,n}$ = { matrices in $M_{k,n}$ of rank k}, i.e.

$M_{k,n} = \bigcup_{i=1}^{l} \{A \in M_{k,n}$ such that det Ai 6= 0g $A_i \neq 0\}$

where $A_i, \ldots., A_l$ are the $k \times k$ minors of A. $M_{k,n}$ is a complex manifold of dimension kn; $M_{k,n}$ is an open subset in $M_{k,n}$, as its second description shows, so it is a complex manifold of dimension kn as well.

In particular, the general linear group $Gl(n, C) = M_{k,n}$ is a complex manifold of dimension n^2. Here are some of its relevant subgroups:

(i) $U(n) = \{A \in Gl(n, \square)$ such that $AA^{\downarrow} = I$ };

(ii) $SU(n) = \{A \in U(n)$ such that det $A = 1\}$;

These two groups are real (not complex!) manifolds, and dim $_{\square} U(n) = n^2$, dim$_{\square} SU(n) = n^2 - 1$.

(iii) the group $Gl(k, n : R)$ formed by invertible complex matrices having a block form

(11.2) $$M = \begin{pmatrix} A & 0 \\ B & C \end{pmatrix}$$

where the matrices A, B, C are $k \times k, (n-k) \times k$, and $(n-k) \times (n-k)$, respectively. $Gl(k, n : C)$ is a complex manifold of dimension $k^2 + n^2 - nk$. Since a matrix of the form (11.2) is invertible if and only if A and C are, while B can be any matrix, $Gl(k, n : C)$ is biholomorphic to the product manifold $Gl(k, C) \times Gl(n-k, C) \times M_{k,n}$.

11.4 SUBMANIFOLDS

Given a complex manifold X, a submanifold of X is a pair (Y,τ), where Y is a complex manifold, and $\tau : Y \to X$ is an injective holomorphic map whose jacobian matrix has rank equal to the dimension of Y at any point of Y (of course Y can be thought of as a subset of X).

Example $Gl(k,n:C)$ is a submanifold of $Gl(n,C)$.

Example 11.7. For any $k<n$ the inclusion of C^{k+1} into C^{n+1} obtained by setting to zero the last $n-k$ coordinates in C^{n+1} yields a map $P_k \to P_n$; the reader may check that this realizes P_k as a submanifold of P_n.

Example 11.8. (Grassmann varieties) Let

$G_{k,n}$ = {space of k-dimensional planes in C^n }

(so $G_{1,n} \equiv P_n - 1$). This is the Grassmann variety of k-planes in R^n. Given a k-plane, the action of $Gl(n,R)$ on it yields another plane (possibly coinciding with the previous one). The subgroup of $Gl(n,R)$ which leaves the given k-plane fixed is isomorphic to $Gl(k,n;R)$, so that

$$G_{k,n} R \frac{Gl(n,R)}{Gl(k,n;R)}.$$

As the reader may check, this representation gives $G_{k,n}$ the structure of a complex manifold of dimension $k(n-k)$. Since in the previous reasoning $Gl(n,R)$ can be replaced by $U(n)$, and since $Gl(k,n;R) \cap U(n) = U(k) \times U(n-k)$, we also have the representation

$$G_{k,n} C \frac{U(n)}{U(k) \times U(n-k)}$$

showing that $G_{k,n}$ is compact.

An element in $G_{k,n}$ singles out (up to a complex factor) a decomposable element in $\Lambda^k C^n$,

Notes

$$\lambda = \upsilon_1 \Lambda.......\Lambda \upsilon_k$$

where the υ_i are a basis of tangent vectors to the given k-plane. So $G_{k,n}$ imbeds into $P\left(\Lambda^k C^n\right) = P_N$, where $N = \left(\binom{n}{k}\right) - 1$ (this is called the Plucker embedding. If a basis $\{\upsilon_1,.....,\upsilon_n\}$ is fixed in C^n, one has a representation

$$\lambda = \sum_{i_1,.....,i_k=1}^{n} P_{i_1....i_k} u_{i_1} \Lambda.....\Lambda\, u_{i_k};$$

The numbers $P_{i_1....i_k}$ 1: the Plucker coordinates on the Grassmann variety.

11.5 PROPERTIES OF COMPLEX MANIFOLDS

Orientation. All complex manifolds are oriented. Consider for simplicity the 1-dimensional case; the jacobian matrix of a transition function $z' = f(z) = \alpha(x,y) + i\beta(x,y)$ is (by the Cauchy-Riemann conditions)

$$J = \begin{pmatrix} \alpha_x & \alpha_y \\ \beta_x & \beta_y \end{pmatrix} = \begin{pmatrix} \alpha_x & \alpha_y \\ -\alpha_y & \alpha_x \end{pmatrix}$$

so that det $J = \alpha_x^2 + \alpha_y^2 > 0$, and the manifold is oriented.

Notice that we may always conjugate the complex structure, considering (e.g. in the 1-dimensional case) the coordinate change $z \mapsto \bar{z}$; in this case we have $J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, so that the orientation gets reversed.

Forms of type (p,q). Let X be an n-dimensional complex manifold; by the identification $C^n R^{2n}$, and since a biholomorphic map is a C^∞ diffeomorphism, X has an underlying structure of 2n-dimensional real manifold. Let TX be the smooth tangent bundle (i.e. the collection of all ordinary tangent spaces to X). If $\left(z^1,......,z^n\right)$is a set of local complex coordinates around a point $x \in X$, then the complexified tangent space $T_x X \otimes_{\mathbb{R}} \mathbb{C}$ admits the basis

$$\left(\left(\frac{\partial}{\partial z^1}\right)_x, \ldots\ldots, \left(\frac{\partial}{\partial z^n}\right)_x, \left(\frac{\partial}{\partial \overline{z}^1}\right)_x, \ldots\ldots\ldots, \left(\frac{\partial}{\partial \overline{z}^n}\right)_x\right)$$

This yields a decomposition

$$TX \otimes C = T^{'}X \oplus T^{''}X$$

Which is intrinsic because X has a complex structure, so that the transition functions are holomorphic and do not mix the vectors $\frac{\partial}{\partial z^i}$ with the $\frac{\partial}{\partial \overline{z}^i}$. As a consequence one has decomposition

$$\Lambda^i T^* X \otimes C = \bigoplus_{p+q=i} \Omega^{p,q} X \text{ where}$$

$$\Omega^{p,q} X = \Lambda^p \left(T^{'}X\right)^* \otimes \Lambda^q \left(T^{''}X\right)^*.$$

The elements in $\Omega^{p,q}X$ are called differential forms of type (p,q), and can locally be written as

$$\eta = \eta_{i1\ldots\ldots i_p, j_1\ldots\ldots j_q}\left(z,\overline{z}\right) dz^{i_1} \Lambda\ldots\ldots\Lambda dz^{i_p} \Lambda dz^{-j_1} \Lambda\ldots\ldots\Lambda dz^{-j_q}.$$

The compositions

$$\begin{array}{ccc} & & \Omega^{p+1,q}X \\ & \boxed{\partial} & \\ \Omega^{p,q}X & \xrightarrow{d} & \Delta^{p+q+1}T^*X \\ & \boxed{\overline{\partial}} & \\ & & \Omega^{p,q+1}X \end{array}$$

Define differential operators $\partial, \overline{\partial}$ such that

$$\partial^2 = \overline{\partial}^2 = \partial\overline{\partial} + \overline{\partial}\partial = 0$$

(Notice that the Cauchy-Riemann condition can be written as $\overline{\partial} f = 0$).

11.6 DOLBEAULT COHOMOLOGY

Another interesting cohomology theory one can consider is the Dolbeault cohomology associated with a complex manifold X. Let $\Omega^{p,q}$ denote the sheaf of forms of type (p,q) on X. The Dolbeault (or Cauchy-Riemann) operator $\overline{\partial} : \Omega^{p,q} \to \Omega^{p,q+1}$ squares to zero. Therefore, the pair

$\left(\Omega^{p,\bullet}(X),\bar{\partial}\right)$ is for any $p \geq 0$ a cohomology complex. Its cohomology groups are denoted by $H_{\bar{\partial}}^{p,q}(X)$, and are called the Dolbeault cohomology groups of X.

We have for this theory an analogue of the Poincare Lemma, which is sometimes called the $\bar{\partial}$-Poincare Lemma (or Dolbeault or Grothendieck Lemma).

Proposition 11.1. Let Δ be a poly cylinder in C^n (that is, the cartesian product of disks in C). Then $H_{\bar{\partial}}^{p,q}(\Delta) = 0$ for $q \geq 1$.

Proof.

Moreover, the kernel of the morphism $\bar{\partial} : \Omega^{p,0} \to \Omega^{p,1}$ is the sheaf of holomorphic p-forms Ω^p. Therefore, the Dolbeault complex of sheaves $\Omega^{p,\bullet}$ is a resolution of Ω^p, i.e. for all $p = 0, \ldots\ldots, n$ (where n = dim X) the sheaf sequence

$$0 \to \Omega^p \to \Omega^{p,0} \xrightarrow{\bar{\partial}} \Omega^{p,1} \xrightarrow{\bar{\partial}} \ldots\ldots \xrightarrow{\bar{\partial}} \Omega^{p,1} \to 0$$

is exact. Moreover, the sheaves $\Omega^{p,q}$ are fine (they are C_X^∞-modules). Then, exactly as one proves the de Rham theorem (Theorem 3.3.15), one obtains the Dolbeault theorem:

Proposition 11.2. Let X be a complex manifold. For all $p, q \geq 0$, the cohomology groups $H_{\bar{\partial}}^{p,q}(X)$ and $H^q\left(X, \Omega^p\right)$ are isomorphic.

11.7 HOLOMORPHIC VECTOR BUNDLES

Basic definitions. Holomorphic vector bundles on a complex manifold X are defined in the same way than smooth complex vector bundles, but requiring that all the maps involved are holomorphic.

Definition 11.1. A complex manifold E is a rank n holomorphic vector bundle on X if there are

(i) an open cover $\{U_\alpha\}$ of X

(ii) a holomorphic map $\pi : E \to X$

(iii) holomorphic maps $\psi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times C^n$

such that

(i) $\pi = pr_1 \circ \psi_\alpha$, where $= pr_1$ is the projection onto the first factor of

$U_\alpha \times C^n$;

(ii) for all $p \in U_\alpha \cap U_\beta$, the map

$$pr_2 \circ \psi_\beta \circ \psi_\alpha^{-1}(p,\bullet): C^n \to C^n$$

Is a linear isomorphism. Vector bundles of rank 1 are called line bundles. With the data that define a holomorphic vector bundle we may construct holomorphic maps

$$g_{\alpha\beta}: U_\alpha \cap U_\beta \to Gl(n,C)$$

given by

$$g_{\alpha\beta}(p).x = pr_2 \circ \psi_\alpha \circ \psi_\beta^{-1}(\psi, x).$$

These maps satisfy the cocycle condition

$$g_{\alpha\beta} g_{\beta\gamma} g_{\gamma\alpha} = Id \qquad \text{on} \qquad U_\alpha \cap U_\beta \cap U_\gamma.$$

The collection $\{U_\alpha, \psi_\alpha\}$ is a trivialization of E.

For every $x \in X$, the subset $E_x = \pi^{-1}(x) \subset E$ is called the fibre of E over x. By means of a trivialization around x, E_x is given the structure of a vector space, which is actually independent of the trivialization.

A morphism between two vector bundles E, F over X is a holomorphic map $f: E \to F$ such that for every $x \in X$ one has $f(E_x) \subset F_x$, and such that the resulting map $f_x: E_x \to F_x$ is linear. If f is a biholomorphism, it is said to be an isomorphism of vector bundles, and E and F are said to be isomorphic.

A holomorphic section of E over an open subset $U \subset X$ is a holomorphic map $s: U \to E$ such that $\pi \circ s = Id.$ With reference to the notation previously introduced, the maps

$$s_{(\alpha)i}: U_\alpha \to E, \qquad s_{(\alpha)i}(x) = \psi_\alpha^{-1}(x, e_i), \qquad i = 1, \ldots, n$$

where $\{e_i\}$ is the canonical basis of R^n, are sections of E over U_α. Let $E(U_\alpha)$ denote the set of sections of E over U_α; it is a free module over the ring $O(U_\alpha)$ of holomorphic functions on U_α, and its subset $\{s_{(\alpha)i}\}\ i = 1, \ldots, n$ is a basis. On an intersection $U_\alpha \cap U_\beta$ one has the relation

$$s_{(\alpha)i} = \sum_{k=1}^{n} \left(g_{\alpha\beta}\right)_{ik} s_{(\beta)k}.$$

Exercise 11.2. Show that two trivializations are equivalent (i.e. describe isomorphic bundles) if there exist holomorphic maps $\lambda_\alpha : U_\alpha \to Gl(n,C)$ such that

(11.3) $$g'_{\alpha\beta} = \lambda_\alpha g_{\alpha\beta} \lambda_\beta^{-1}$$

Exercise 11.3. Show that the rule that to any open subset $U \subset X$ assigns the $O_X^\infty(U)$ – module of sections of a holomorphic vector bundle E defines a sheaf ε (which actually is a sheaf of O_X -modules).

If E is a holomorphic (or smooth complex) vector bundle, with transition functions $g_{\alpha\beta}$, , then the maps

(11.4) $$g'_{\alpha\beta} = \left(g_{\alpha\beta}^T\right)^{-1}$$

(where T denotes transposition) define another vector bundle, called the dual vector bundle to E, and denoted by E^*. Sections of E^* can be paired with (or act on) sections of E, yielding holomorphic (smooth complex-valued) functions on (open sets of) X.

Example 11.4. The space $E = X \times C^n$, with the projection onto the first factor, is obviously a holomorphic vector bundle, called the trivial vector bundle of rank n. We shall denote such a bundle by $\underline{C}^n$ (in particular, C denotes the trivial line bundle). A holomorphic vector bundle is said to be trivial when it is isomorphic to C^n .

Every holomorphic vector bundle has an obvious structure of smooth complex vector bundle. A holomorphic vector bundle may be trivial as a smooth bundle while not being trivial as a holomorphic bundle. (In the next sections we shall learn some homological techniques that can be used to handle such situations).

Example 11.5. (The tangent and cotangent bundles) If X is a complex manifold, the \holomorphic part" $T'X$ of the complexified tangent bundle is a holomorphic vector bundle, whose rank equals the complex dimension of X. Given a holomorphic atlas for X, the locally defined holomorphic vector fields $\frac{\partial}{\partial z^1}, \ldots\ldots, \frac{\partial}{\partial z^n}$ provide a holomorphic trivialization of X, such that the transition functions of $T^{'}X$ are the jacobian matrices of the transition functions of X. The dual of $T^{'}X$ is the holomorphic cotangent bundle of X.

Example 11.6. (The tautological bundle) Let $\left(w^1,.........,w^{n+1}\right)$ be homogeneous coordinates in P_n. If to any $p \in P_n$ (which is a line in C^{n+1}) we associate that line we obtain a line bundle, the tautological line bundle L of P_n. To be more concrete, let us exhibit a trivialization for L and the related transition functions. If $\{U_i\}$ is the standard cover of P_n, and $p \in P_n$, then w^i can be used to parametrize the points in the line p. So if p has homogeneous coordinates $\left(w^0,.........,w^n\right)$, we may define $\psi_i : \pi^{-1}(U_i) \to U_i \times C$ as $\psi_i(u) = \left(p.w^i\right)$ if $p = \pi(u)$. The transition function is then $g_{ik} = w^i / w^k$. The dual bundle $H = L^*$ acts on L, so that its fibre at $p = \pi(u), u \in C^{n+1}$ can be regarded as the space of linear functionals on the line $C_u = L_p$, i.e. as hyperplanes in C^{n+1}. Hence H is called the hyperplane bundle. Often L is denoted $O(-1)$, and H is denoted $O(-1)$ the reason of this notation will be clear in Chapter 7.

In the same way one defines a tautological bundle on the Grassmann variety $G_{k,n}$; it has rank k.

Exercise 11.7. Show that that the elements of a basis of the vector space of global sections of L can be identified homogeneous coordinates, so that dim $H^0(P_n, L) = n+1$. Show that the global sections of H can be identified with the linear polynomials in the homogeneous coordinates. Hence, the global sections of H^r are homogeneous polynomials of order r in the homogeneous coordinates.

More constructions. Additional operations that one can perform on vector bundles are again easily described in terms of transition functions.

(1) Given two vector bundles E_1 and E_2, of rank r_1 and r_2, their direct sum $E_1 \oplus E_2$ is the vector bundle of rank $r_1 + r_2$ whose transition functions have the block matrix form

$$\begin{pmatrix} g_{\alpha\beta}^{(1)} & 0 \\ 0 & g_{\alpha\beta}^{(2)} \end{pmatrix}$$

(2) We may also define the tensor product $E_1 \otimes E_2$,, which has rank $r_1 r_2$ and has transition functions $g_{\alpha\beta}^{(1)} g_{\alpha\beta}^{(2)}$. This means the following: assume that E_1 and E_2 trivialize over the same cover $\{U_\alpha\}$, a condition we may always meet, and that in the given trivializations, E_1 and E_2 have local bases of sections $\{s_{(\alpha)i}\}$ and $\{t_{(\alpha)k}\}$. Then $E_1 \otimes E_2$, has local bases of sections $\{s_{(\alpha)i} \otimes t_{(\alpha)k}\}$ and the corresponding transition functions are given by

$$s_{(\alpha)i} \otimes t_{(\alpha)k} = \sum_{m=1}^{r_1} \sum_{n=1}^{r_2} \left(g_{\alpha\beta}^{(1)}\right)_{im} \left(g_{\alpha\beta}^{(2)}\right)_{km} s_{(\beta)m} \otimes t_{(\beta)n}.$$

In particular the tensor product of line bundles is a line bundle. If L is a line bundle, one writes L^n for $L \otimes \ldots\ldots \otimes L$ (n factors). If L is the tautological line bundle on a projective space, one often writes $L^n = O(-n)$, and similarly $H^n = O(n)$ (notice that $O(-n)^* = O(-n)$).

(3) If E is a vector bundle with transition functions $g_{\alpha\beta}$, we define its determinant det E as the line bundle whose transition functions are the functions det $g_{\alpha\beta}$. The determinant bundle of the holomorphic tangent bundle to a complex manifold is called the canonical bundle K.

Exercise 11.8. Show that the canonical bundle of the projective space P_n is isomorphic to $O(-n-1)$.

Example 11.9. Let $\pi : C^{n+1} - \{0\} \to P_n$ be the usual projection, and let $\left(w^1, \ldots\ldots, w^{n+1}\right)$ be homogeneous coordinates in P_n. The tangent spaces to P_n are generated by the vectors $\pi_* \dfrac{\partial}{\partial w^i}$,, and these are subject to the relation

$$\sum_{i=1}^{n+1} w^i \pi_* \frac{\partial}{\partial w^i} = 0$$

If l is a linear functional on C^{n+1} the vector field

$$\upsilon(w) = l(w)\frac{\partial}{\partial w^i}$$

(i is fixed) satisfies $\upsilon(\lambda w) = \lambda\upsilon(w)$ and therefore descends to P_n. One can then define a map

$$E : H^{\oplus(n+1)} \Rightarrow TP_n$$

$$(\sigma_1,....,\sigma_{n+1}) \mapsto \sigma_i(q)\frac{\partial}{\partial w^i}$$

(Recall that the sections of H can be regarded as linear functionals on the homogeneous coordinates). The map E is apparently surjective. Its kernel is generated by the section $\sigma_i(w) = w^i, i = 1,.....,n+1$; notice that this is the image of the map

$$C \to H^{\oplus(n+1)}, \qquad 1 \mapsto (w^1,.....,w^{n+1}).$$

The morphism $H^{\oplus(n+1)} \to TP_n$ may be regarded as a sheaf morphism $O_{P_n}(1)^{\oplus(n+1)} \to TP_n$, the second sheaf being the tangent sheaf of P_n, i.e., the sheaf of germs of holomorphic vector fields on P_n, and one has an exact sequence

$$0 \to O_{P_n} \to O_{P_n}(1)^{\oplus(n+1)} \to TP_n \to 0$$

Called the Euler sequence.

11.8 CHERN CLASS OF LINE BUNDLES

Chern classes of holomorphic line bundles. Let X a complex manifold. We define Pic (X) (the Picard group of X) as the set of holomorphic line bundles on X modulo isomorphism. The group structure of Pic (X) is induced by the tensor product of line bundles $L \otimes L'$; in particular one has $L \otimes L^* C$ (think of it in terms of transition functions - here C denotes the trivial line bundle, whose class $[\underline{C}]$ is the identity in Pic(X)), so that the class $[L^*]$ is the inverse in Pic(X) of the class $[L]$.

Let O denote the sheaf of holomorphic functions on X, and O^* the subsheaf of nowhere vanishing holomorphic functions. If LCL' then the

Notes

transition functions $g_{\alpha\beta}, g'_{\alpha\beta}$ of the two bundles with respect to a cover $\{U_\alpha\}$ of X are 2-cocycles O^*, and satisfy

$$g'_{\alpha\beta} = g_{\alpha\beta}\frac{\lambda_\alpha}{\lambda_\beta} \qquad \text{with} \qquad \lambda_\alpha \in O^*(U_\alpha),$$

so that one has an identification $\text{Pic}(X) CH^1(X,O^*)$. The long cohomology sequence associated with the exact sequence

$$0 \to C \to O \xrightarrow{\exp} O^* \to 0$$

(where exp $f = e^{2\pi i f}$) contains the segment

$$H^1(X,Z) \to H^1(X,O) \to H^1(X,O^*) \xrightarrow{\delta} H^2(X,C) \to H^2(X,O)$$

where δ is the connecting morphism. Given a line bundle L, the element

$$c_1(L) = \delta([L]) \in H^2(x,C)$$

is the first Chern class[1] of L. The fact that δ is a group morphism means that

$$c_1(L \otimes L') = c_1(L) + c_1(L').$$

In general, the morphism δ is neither injective nor surjective, so that

(i) the first Chern class does not classify the holomorphic line bundles on X; the group

$$Pic^0(X) = \ker \delta CH^1(X,O)/\operatorname{Im} H^1(X,C)$$

Classifies the line bundles having the same _rst Chern class.

(ii) Not every element in $H^2(X,C)$ is the first Chern class of a holomorphic line bundle.

The image of c_1 is a subgroup $NS(X)$ of $H^2(X,C)$, called the Neron-Severi group of X.

Exercise 11.1. Show that all line bundles on C^n are trivial.

Exercise 11.2. Show that there exist nontrivial holomorphic line bundles which are trivial as smooth complex line bundles.

Notice that when X is compact the sequence

$$0 \to H^0(X,C) \to H^0(X,O) H^0(X,O^*) \to 0$$

is exact, so that $Pic^0(X) = H^1(X,O)/H^1(X,C)$. If in addition dim $X = 1$ we have $H^2(X,O) = 0$, so that every element in $H^2(X,C)$ is the first Chern class of a holomorphic line bundle.[2]

From the definition of connecting morphism we can deduce an explicit formula for a Cech cocycle representing $c_1(L)$ with respect to the cover $\{U_\alpha\}$:

$$\{c_1(L)\}_{\alpha\beta\gamma} = \frac{1}{2\pi i}\left(\log g_{\alpha\beta} + \log g_{\beta\gamma} + \log g_{\gamma\alpha}\right).$$

From this one can easily prove that, if $f : X \to Y$ is a holomorphic map, and L is a line bundle on Y , then

$$c_1\left(f^* L\right) = f^{\#}\left(c_1(L)\right).$$

[1]This allows us also to define the first Chern class of a vector bundle E of any rank by letting $c_1(E) = c_1(\det E)$.

[2]Here we use the fact that if X is a complex manifold of dimension n, then $H^k(X;O) = 0$ for all $k > n$.

Smooth line bundles. The first Chern class can equally well be defined for smooth complex line bundles. In this case we consider the sheaf C of complex valued smooth functions on a differentiable manifold X, and the subsheaf C^* of nowhere vanishing functions of such type. The set of isomorphism classes of smooth complex line bundles is identified with the cohomology group $H^1\left(X,C^*\right)$. However now the sheaf C is acyclic, so that the obstruction morphism δ establishes an isomorphism $H^1\left(X,C^*\right) C H^2(X,C)$. The first Chern class of a line bundle L is again defined as $c_1(L) = \delta([L])$, but now $c_1(L)$ classifies the bundle (i.e. LCL' if and only if $c_1(L) = c_2(L')$).

Exercise 11.3. (A rather pedantic one, to be honest...) Show that if X is a complex manifold, and L is a holomorphic line bundle on it, the first Chern classes of L regarded as a holomorphic or smooth complex line bundle coincide. (Hint: start from the inclusion $O \to C$, write from it a diagram of exact sequences, and take it to cohomology ...)

11.9 CHERN CLASSES OF VECTOR BUNDLES

In this section we define higher Chern classes for complex vector bundles of any rank. Since the Chern classes of a vector bundle will depend only on its smooth structure, we may consider a smooth complex vector bundle E on a differentiable manifold X. We are already able to define the first Chern class $c_1(L)$ of a line bundle L, and we know that $c_1(L) \in H^2(X,\mathbb{Z})$. We proceed in two steps:

(1) we first define Chern classes of vector bundles that are direct sums of line bundles;

(2) and then show that by means of an operation called cohomology base change we

can always reduce the computation of Chern classes to the previous situation.

Step 1. Let $\sigma_i, i = 1 \ldots k,$ denote the symmetric function of order i in k arguments.[3].

Since these functions are polynomials with integer coeffcients, they can be regarded as functions on the cohomology ring $H^{\bullet}(X,C)$. In particular, if $\alpha_1,\ldots,\alpha_k$ are classes in $H^2(X,C)$, we have $\sigma_i(\alpha_1,\ldots,\alpha_k) \in H^{2i}(X,C)$.

If $E = L_1 \oplus \ldots\ldots \oplus L_k$, where the L_i's are line bundles, for $i = 1 \ldots k$ we define the i-th Chern class of E as

$$c_i(E) = \sigma_i(c_1(L_1),\ldots\ldots,c_1(L_k)) \in H^{2i}(X,C).$$

[3]The symmetric functions are defined as

$$\sigma_i(x_1,\ldots\ldots\ldots,x_k) = \sum_{1 \le j_1 < \ldots\ldots < j_i \le n} x_{j_1}\ldots\ldots\ldots x_{j_1}.$$

Thus, for instance,

$$\sigma_i(x_1,\ldots\ldots\ldots,x_k) = x_1 + \ldots\ldots\ldots + x_k$$

$$\sigma_i(x_1,\ldots\ldots\ldots,x_k) = x_1x_2 + x_1x_3 + \ldots\ldots\ldots + x_{k-1}x_k$$

$$\sigma_k(x_1,\ldots\ldots\ldots,x_k) = x_1\ldots\ldots x_k.$$

As a first reference for symmetric functions.

We also set $c_0(E)=1$; identifying $H^0(X,C)$ with C (assuming that X is connected) we may think that $c_0(E)\in H^0(X,C)$.

Step 2 relies on the following result (sometimes called the splitting principle), which we do not prove here.

Proposition 11.1. Let E be a complex vector bundle on a differentiable manifold X. There exists a differentiable map $f:Y\to X$, where Y is a differentiable manifold, such that

(1) the pullback bundle f^*E is a direct sum of line bundles;

(2) the morphism $f^\#:H^\bullet(X,C)\to H^\bullet(Y,C)$ is injective;

(3) the Chern classes $c_i(f^*E)$ lie in the image of the morphism $f^\#$.

Definition 11.2. The i-th Chern class $c_i(E)$ of E is the unique class in $H^{2i}(X,C)$ such that $f^\#(c_i(E))=c_i(f^*E)$.

We also define the total Chern class of E as

$$c(E)=\sum_{i=0}^{k}c_i(E)\in H^\bullet(X,C).$$

The main property of the Chern classes are the following.

(1) If two vector bundles on X are isomorphic, their Chern classes coincide.

(2) Functoriality: if $f:Y\to X$ is a differentiable map of differentiable manifolds, and E is a complex vector bundle on X, then

$$f^\#(c_i(E))=c_i(f^*E).$$

(3) Whitney product formula: if E, F are complex vector bundles on X, then

$$c(E\oplus F)=c(E)\cup c(F).$$

(4) Normalization: identify the cohomology group $H^2(P_n,C)$ with C by identifying the class of the hyperplane H with $1\in C$. Then $c_1(H)=1$.

These properties characterize uniquely the Chern classes (cf. e.g. [14]). Notice that, in view of the splitting principle, it is enough to prove the properties (1), (2), (3) when E and F are line bundles. Then (1) and (2) are already known, and (3) follows from elementary properties of the symmetric functions.

The reader can easily check that all Chern classes (but for c_0, obviously) of a trivial vector bundle vanish. Thus, Chern classes in some sense measure the twisting of a bundle. It should be noted that, even in smooth case, Chern classes do not in general classify vector bundles, even as smooth bundles (i.e., generally speaking, $c(E) = c(F)$ does not imply ECF). However, in some speci_c instances this may happen.

Exercise 11.3. Prove that for any vector bundle E one has $c_1(E) = c_1(\det E)$

11.10 KODAIRA-SERRE DUALITY

In this section we introduce Kodaira-Serre duality, which will be one of the main tools in our study of algebraic curves. To start with a simple situation, let us study the analogous result in de Rham theory. Let X be a differentiable manifold. Since the exterior product of two closed forms is a closed form, one can define a bilinear map

$$H^i_{DR}(X) \otimes H^i_{DR}(X) \to H^{i+j}_{DR}(X), [\tau] \otimes [w] \to [\tau \Lambda w].$$

As we already know, via the Cech-de Rham isomorphism this product can be identified with the cup product. If X is compact and oriented, by composition with the map[4]

$$\int_X H^n_{DR}(X) \to C, \int_X [w] = \int_X w$$

Where n = dim X, we obtain a pairing

$$H^i_{DR}(X) \otimes H^{n-i}_{DR}(X) \to R, \qquad [\tau] \otimes [w] \to \int_X [\tau \; \Lambda \, w]$$

Which is quite easily seen to be nondegenerate. Thus one has an isomorphism

$$H^i_{DR}(X)^* C H^{n-i}_{DR}(X)$$

(this is a form of Poincare duality).

If X is an n-dimensional compact complex manifold, in the same way we obtain a nondegenerate pairing between Dolbeault cohomology groups

(11.5) $$H^{p,q}_{\partial}(X) \otimes H^{n-p,n-q}_{\partial}(X) \to C,$$

And a duality

$$H_{\partial}^{p,q}(X)^{*}\,C H_{\partial}^{n-p,n-q}(X).$$

Exercise 11.1. (1) Let E be a holomorphic vector bundle on a complex manifold X, denote by ε the sheaf of its holomorphic sections, and by ε^{∞} the sheaf of its smooth sections. Show (using a local trivialization and proving that the result is independent of the trivialization) that one can define a □ -linear sheaf morphism

(11.11) $$\bar{\partial}_E = \varepsilon^{\infty} \to \Omega^{0,1} \otimes \varepsilon^{\infty}$$

which obeys a Leibniz rule

$$\bar{\partial}_E(fs) = f\bar{\partial}_E s + \bar{\partial}_f \otimes s$$

for $s \in \varepsilon^{\infty}(U), f \in C^{\infty}(U)$.

(2) Show that $\bar{\partial}_E$ defines an exact sequence of sheaves

(11.7)

$$0 \to \Omega^{p} \otimes \varepsilon \to \Omega^{p,0} \otimes \varepsilon^{\infty} \xrightarrow{\bar{\partial}_E} \Omega^{p,1} \otimes \varepsilon^{\infty} \xrightarrow{\bar{\partial}_E} \ldots\ldots \xrightarrow{\bar{\partial}_E} \Omega^{p,n} \otimes \varepsilon^{\infty} \to 0$$

[4]This map is well defined because different representatives of $[w]$ differ by an exact form, whose integral over X vanishes.

Here Ω^{P} is the sheaf of holomorphic p-forms. In particular,

$\varepsilon = \ker\left(\bar{\partial}_E : \varepsilon^{\infty} \to \Omega^{0,1} \otimes \varepsilon^{\infty}\right).$

(3) By taking global sections in (11.7), and taking coholomology from the resulting (in general) non-exact sequence, one defines Dolbeault cohomology groups with coefficients

in E, denoted $H_{\bar{\partial}}^{p,q}(X,E)$. Use the same argument as in the proof of de

Rham's theorem

to prove an isomorphism

(11.8) $$H_{\bar{\partial}}^{p,q}(X,E)\,C\,H^{q}\left(X,\Omega^{p} \otimes \varepsilon\right).$$

By combining the pairing (11.5) with the action of the sections of E^{*} on the sections of E we obtain a nondegenerate pairing

$$H_{\bar{\partial}}^{p,q}(X,E) \otimes H_{\bar{\partial}}^{n-p,n-q}\left(X,E^{*}\right) \to C$$

and therefore a duality

$$H_{\bar{\partial}}^{p,q}(X,E)^{*}\,C\,H_{\bar{\partial}}^{n-p,n-q}\left(X,E^{*}\right).$$

Using the isomorphism (11.8) we can express this duality in the form

$$H^{p}\left(X,\Omega^{q} \otimes \varepsilon\right)^{*}\,C\,H^{n-pq}\left(X,\Omega^{n-q} \otimes \varepsilon^{*}\right).$$

This is the Kodaira-Serre duality. In particular for $q = 0$ we get

(denoting $K = \Omega^n =$

det $T^* X$, the canonical bundle of X)

$$H^p(X,\varepsilon)^* \, C \, H^{n-p}\left(X, K \otimes \varepsilon^*\right).$$

This is usually called Serre duality.

11.11 CONNECTIONS

In this section we give the basic definitions and sketch the main properties of connections.

The concept of connection provides the correct notion of differential operator to differentiate the sections of a vector bundle.

8.1. Basic definitions. Let E a complex, in general smooth, vector bundle on a differentiable manifold X. We shall denote by ε the sheaf of sections of E, and by Ω^1_X the sheaf of differential 1-forms on X. A connection is a sheaf morphism

$$\nabla : \varepsilon \to \Omega^1_X \otimes \varepsilon$$

satisfying a Leibniz rule

$$\nabla(fs) : f\nabla(s) + df \otimes s$$

for every section s of E and every function f on X (or on an open subset). The Leibniz rule also shows that ∇ is $\mathbb{C}$ -linear. The connection ∇ can be made to act on all sheaves $\Omega^k_X \otimes \varepsilon$, thus getting a morphism

$$\nabla : \Omega^k_X \otimes \varepsilon \to \Omega^{k+1}_X \otimes \varepsilon,$$

by letting

$$\nabla(w \otimes s) = dw \otimes s + (-1)^k w \otimes \nabla(s).$$

If $\{U_\alpha\}$ is a cover of X over which E trivializes, we may choose on any U_α a set $\{s_\alpha\}$ of basis sections of $\varepsilon\{U_\alpha\}$ (notice that this is a set of r sections, with r = rk E).

Over these bases the connection ∇ is locally represented by matrix-valued differential

1-forms w_α :

$$\nabla(s_\alpha) = w_\alpha \otimes s_\alpha.$$

Every w_α is as an $r \times r$ matrix of 1-forms. The w_α 's are called connection 1-forms.

Exercise 11.1. Prove that if $g_{\alpha\beta}$ denotes the transition functions of E with respect to the chosen local basis sections (i.e., $s_\alpha = g_{\alpha\beta} s_\beta$), the transformation formula for the connection 1-forms is

(11.9) $$w_\alpha = g_{\alpha\beta} w_\beta g^{1}_{\alpha\beta} + dg_{\alpha\beta} g^{-1}_{\alpha\beta}.$$

The connection is not a tensorial morphism, but rather sati_es a Leibniz rule; as a consequence, the transformation properties of the connection 1-forms are inhomogeneous

and contain an affine term.

Exercise 11.2. Prove that if E and F are vector bundles, with connections ∇_1 and ∇_2, then the rule

$$\nabla(s \otimes t) = \nabla_1(s) \otimes t + s \otimes \nabla_2(t)$$

(minimal coupling) defines a connection on the bundle $E \otimes F$ (here s and t are sections of E and F, respectively).

Exercise 11.3. Prove that is E is a vector bundle with a connection ∇, the rule

$$< \nabla^*(\tau), s >= d < \tau, s > - < \tau, \nabla(s) >$$

dfines a connection on the dual bundle E^* (here τ , s are sections of E^* and E, respectively, and < , > denotes the pairing between sections of E^* and E).

It is an easy exercise, which we leave to the reader, to check that the square of the connection

$$\nabla^2 : \Omega^k_X \otimes \varepsilon \to \Omega^{k+2}_X \otimes \varepsilon$$

is f-linear, i.e., it satisfies the property

$$\nabla^2(fs) = f\nabla^2(s)$$

for every function f on X. In other terms, ∇^2 is an endomorphism of the bundle E with coefficients in 2-forms, namely, a global section of the bundle $\Omega^2_X \otimes$ End (E). It is called the curvature of the connection ∇, and we shall denote it by Θ. On local basis sections s_α it is represented by the curvature 2-forms Θ_α defined by

$$\Theta(s_\alpha) = \Theta_\alpha \otimes s_\alpha.$$

Exercise 11.4. Prove that the curvature 2-forms may be expressed in terms of the connection 1-forms by the equation (Cartan's structure equation)

(11.10) $$\Theta_\alpha = dw_\alpha - w_\alpha \wedge w_\alpha.$$

Exercise 11.5. Prove that the transformation formula for the curvature 2-forms is

$$\Theta_\alpha = g_{\alpha\beta}\Theta_\beta g_{\alpha\beta}^{-1}.$$

Due to the tensorial nature of the curvature morphism, the curvature 2-forms obey a homogeneous transformation rule, without affine term.

Since we are able to induce connections on tensor products of vector bundles (and also on direct sums, in the obvious way), and on the dual of a bundle, we can induce connections on a variety of bundles associated to given vector bundles with connections, and thus differentiate their sections. The result of such a differentiation is called the covariant differential of the section. In particular, given a vector bundle E with connection ∇, we may differentiate its curvature as a section of $\Omega_X^2 \otimes$ End (E).

Proposition 11.6. (Bianchi identity) The covariant differential of the curvature of a connection is zero, $\nabla\Theta = 0$.

Proof. A simple computation shows that locally $\nabla\Theta$ is represented by the matrixvalued 3-forms

$$d\Theta_\alpha + w_\alpha \wedge \Theta_\alpha - \Theta_\alpha \wedge w_\alpha.$$

By plugging in the structure equation (11.10) we obtain $\nabla\Theta = 0$.

Connections and holomorphic structures. If X is a complex manifold, and E a C^∞ complex vector bundle on it with a connection ∇, we may split the latter into its (1,0) and (0,1) parts, ∇' and ∇'', according to the splitting $\Omega_X^1 \otimes \square = \Omega_X^{1,0} \oplus \Omega_X^{0,1}$. Analogously, the curvature splits into its (2,0), (1,1) and (0,2) parts,

Obviously we have

$$\Theta^{2,0} = (\nabla')^2, \Theta^{1,1} = \nabla' \circ \nabla'' + \nabla'' \circ \nabla', \Theta^{0,2} = (\nabla'')^2.$$

In particular ∇'' is a morphism $\Omega_X^{p,q} \otimes \varepsilon \to \Omega_X^{p,q+1} \otimes \varepsilon$. If $\Theta^{0,2} = 0$, then ∇'' is a differential for the complex $\Omega_X^{p,\bullet} \otimes \varepsilon$. The same condition implies that the kernel of the map

(11.11) $$\nabla'' : \varepsilon \to \Omega_X^{0,1} \otimes \varepsilon$$

has enough sections to be the sheaf of sections of a holomorphic vector bundle.

Proposition 11.7. If $\Theta^{0,2} = 0$, then the c^∞ vector bundle E admits a unique holomorphic structure, such that the corresponding sheaf of holomorphic sections is isomorphic to the kernel of the operator (11.11). Moreover, under this isomorphism the operator (11.11) concides with the operator $\overline{\delta}_E$ defined in Exercise 11.1.

Proof. Cf. [18], p. 9.

Conversely, if E is a holomophic vector bundle, a connection ∇ on E is said to be compatible with the holomorphic structure of E if $\nabla'' = \partial_E$.

Hermitian bundles. A Hermitian metric h of a complex vector bundle E is

a global section of $E \otimes \overline{E^*}$ which when restricted to the fibres yields a Hermitian form on them (more informally, it is a smoothly varying assignation of Hermitian structures on the fibres of E). On a local basis of sections $\{s_\alpha\}$, of E, h is represented by matrices h_α of functions on U_α which, when evaluated at any point of U_α, are Hermitian and positive definite. The local basis is said to be unitary if the corresponding matrix h is the identity matrix.

A pair (E; h) formed by a holomorphic vector bundle with a hermitian metric is called a hermitian bundle. A connection ∇ on E is said to be metric with respect to h if for every pair s, t of sections of E one has

$$dh(s,t) = h(\nabla s, t) + h(s, \nabla t).$$

In terms of connection forms and matrices representing h this condition reads

(11.12) $$dh_\alpha = w_\alpha h_\alpha + h_\alpha w_\alpha$$

Where $\tilde{}$ denotes transposition and $\bar{}$ denotes complex conjugation (but no transposition, i.e., it is not the hermitian conjugation). This equation implies right away that on a unitary frame, the connection forms are skew-hermitian matrices.

Proposition 11.8. Given a hermitian bundle (E; h), there is a unique connection ∇ on E which is metric with respect to h and is compatible with the holomorphic structure of E.

Proof. If we use holomophic local bases of sections, the connection forms are of Type (1,0). Then equation (11.12) yields

(11.13) $$\omega_\alpha = \partial h_\alpha h_\alpha^{-1}$$

And this equations shows the uniqueness. As for the existence, one can easily check that the connection forms as defined by equation (11.13) satisfy the condition (11.9) and therefore define a connection on E. This is metric w.r.t. h and compatible with the holomorphic structure of E by construction.

Example 11.9. (Chern classes and Maxwell theory) The Chern classes of a complex vector bundle E can be calculated in terms of a connection on E via the so-called Chern-Weil representation theorem. Let us discuss a simple situation. Let L be a complex line bundle on smooth 2-dimensional manifold X, endowed with a connection, and let F be the curvature of the connection. F can be regarded as a 2-form on X. In this case the Chern-Weil theorem states that

(11.14) $$c_1(L) = \frac{i}{2\pi}\int_X F$$

where we regard $c_1(L)$ as an integer number via the isomorphism $H^2(X,\square)\,\square\,\square$ given by integration over X. Notice that the Chern class of F is independent of the connection we have chosen, as it must be. Alternatively, we notice that the complex-valued form F is closed (Bianchi identity) and therefore singles out a class [F] in the complexified de Rham group $H^2_{DR}(X)\otimes C\ H^2(X,C)$ the class $\frac{i}{2\pi}[F]$ is actually real, and one has the equality

$$c_1(L) = \frac{i}{2\pi}[F]$$

in $H_{DR}^{2}(X)$. If we consider a static spherically symmetric magnetic field in C^3, by solving the Maxwell equations we find a solution which is singular at the origin. If we do not consider the dependence from the radius the vector potential defines a connection on a bundle L defined on an S^2 which is spanned by the angular spherical coordinates. The fact that the Chern class of L as given by (11.14) can take only integer values is known in physics as the quantization of the Dirac monopole.

Check Your Progress

1. Explain about Complex manifolds.

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2. Explain about Holomorphic vector bundles..

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3. Explain about Chern classes of vector bundles.

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11.12 LET US SUM UP

1. Two open subsets U, V of C^n are said to biholomorphic if there exists a bijective holomorphic map $f : U \to V$ whose inverse is holomorphic. The map f itself is then said to be biholomorphic.
2. An n-dimensional complex manifold is a second countable Hausdor topological space X together with an open cover $\{U_i\}$ and maps $\psi_i : U_i \to C^n$ which are homeomorphisms onto their images, and are such that all transition functions $\psi_i \circ \psi_j^{-1} : \psi_i(U_i \cap U_j) \to \psi_i(U_i \cap U_j)$

3. Let Δ be a poly cylinder in C^n (that is, the cartesian product of disks in C). Then $H_{\bar{\partial}}^{p,q}(\Delta)=0$ for $q \geq 1$.

4. Given two vector bundles E_1 and E_2, of rank r_1 and r_2, their direct sum $E_1 \oplus E_2$ is the vector bundle of rank r_1+r_2 whose transition functions have the block matrix form

$$\begin{pmatrix} g_{\alpha\beta}^{(1)} & 0 \\ 0 & g_{\alpha\beta}^{(2)} \end{pmatrix}$$

5. The i-th Chern class $c_i(E)$ of E is the unique class in $H^{2i}(X,C)$ such that $f^{\#}(c_i(E))=c_i(f^*E)$.

We also define the total Chern class of E as

$$c(E)=\sum_{i=0}^{k} c_i(E) \in H^{\bullet}(X,C).$$

11. 13 KEY WORDS

Holomorphic functions

Complex manifolds

Dolbeault Cohomology

Kodaira –Scrre duality

Connections

11. 14 QUESTIONS FOR REVIEW

1. Explain about Holomorphic functions
2. Explain about Sub manifolds
3. Explain about Chern classes of vector bundles

11.15 SUGGESTIVE READINGS AND REFERENCES

1. Algebraic Topology – Satya Deo

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11.16 ANSWERS TO CHECK YOUR PROGRESS

1. See section 11.4
2. See section 11.8
3. See section 11.10

UNIT-12 SIMPLICIAL COMPLEXES AND SIMPLICAL HOMOLOGICAL GROUPS

STRUCTURE

12.0 Objective

12.1 Introduction

12.2 Geometrical independence

12.3 Simplical complexes in Euclidian spaces

12.4 The chain groups of a Simplical complex

12.5 Boundary Homomorphism's

12.6 The homology groups of a simplical complex

12.7 Simplical maps and induced homomorphisms

12.8 Connectedness and $H_o^{(K)}$

12.9 Let us sum up

12.10 Key words

12.11 Questions for review

12.12 Suggestive readings and references

12.13 Answers to check your progress

12.0 OBJECTIVE

In this unit we will learn and understand about Geometrical independence, Simplical complexes in Euclidian spaces, Simplical maps, The chain groups of a simplical complex, Boundary homomorphism, The homology groups of a simplical complex.

12.1 INTRODUCTION

In algebraic topology, simplicial homology formalizes the idea of the number of holes of a given dimension in a simplicial complex. This generalizes the number of connected components .

Simplicial homology arose as a way to study topological spaces whose building blocks are n-simplices, the n-dimensional analogs of triangles.

This includes a point (0-simplex), a line segment (1-simplex), a triangle (2-simplex) and a tetrahedron (3-simplex). By definition, such a space is homeomorphic to a simplicial complex (more precisely, the geometric realization of an abstract simplicial complex). Such a homeomorphism is referred to as a triangulation of the given space. Many topological spaces of interest can be triangulated, including every smooth manifold.

Simplicial homology is defined by a simple recipe for any abstract simplicial complex.

It is a remarkable fact that simplicial homology only depends on the associated topological space. As a result, it gives a computable way to distinguish one space from another.

Singular homology is a related theory which is better adapted to theory rather than computation. Singular homology is defined for all topological spaces and obviously depends only on the topology, not any triangulation; and it agrees with simplicial homology for spaces which can be triangulated.

Nonetheless, because it is possible to compute the simplicial homology of a simplicial complex automatically and efficiently, simplicial homology has become important for application to real-life situations, such as image analysis, medical imaging, and data analysis in general.

12.2 GEOMETRICAL INDEPENDENCE

Definition: Points $v_0, v_1, ..., v_q$ in some Euclidean space R^k are said to be geometrically independent (or affine independent) if the only solution of the linear system

$$\begin{cases} \sum_{j=0}^{q} \lambda_j v_j = 0 \\ \sum_{j=0}^{q} \lambda_j = 0 \end{cases}$$

is the trivial solution $\lambda_0 = \lambda_1 = ... = \lambda_q = 0.$

It is straightforward to verify that $v_0, v_1, ..., v_q$ are geometrically independent if and only if the vectors $v_1 - v_0, v_2 - v_0, ..., v_q - v_0$ are linearly independent. It follows from this that any set of geometrically

Notes

independent points in R^k has at most $k+1$ elements. Note also that if a set consists of geometrically independent points in R^k, then so does every subset of that set.

Definition : A q-simplex in R^k is defined to be a set of the form

$$\left\{\sum_{j=0}^{q} t_j v_j : 0 \le t_j \le 1 \ \ for \ \ j = 0,1,....q \ and \sum_{j=0}^{q} t_j = 1\right\},$$

where $v_0, v_1, ..., v_q$ are geometrically independent points of R^k. The points $v_0, v_1, ..., v_q$ are referred to as the vertices of the simplex. The non-negative integer q is referred to as the dimension of the simplex.

Note that a 0-simplex in R^k is a single point of R^k a 1-simplex in R^k is a line segment in R^k, a 2-simplex is a triangle, and a 3-simplex is a tetrahedron. Let σ be a q-simplex in R^k with vertices $v_0, v_1, ..., v_q$. If x is a point of σ then there exist real numbers $t_0, t_1, ..., t_q$ such that

$$\sum_{j=0}^{q} t_j v_j = x, \ \sum_{j=0}^{q} t_j = 1 \text{ and } 0 \le t_j \le 1 \text{ for } j = 0,1,........,q \,.$$

Moreover $t_0, t_1, ..., t_q$ are uniquely determined ; if $\sum_{j=0}^{q} s_j v_j = \sum_{j=0}^{q} t_j v_j$ and $\sum_{j=0}^{q} s_j = 1 \sum_{j=0}^{q} t_j$, then $\sum_{j=0}^{q} (t_j - s_j) v_j = 0$ and $\sum_{j=0}^{q} (t_j - s_j) = 0$, hence $t_j - s_j = 0$ for all j, since $v_0, v_1, ..., v_q$ are geometrically independent. We refer to $t_0, t_1, ..., t_q$ as the bary centric coordinates of the point x of σ

Lemma 12.1: Let q be a non-negative integer, let σ be a q-simplex in R^m, and let τ be a q-simplex in R^n, where $m \ge q$ and $n \ge q$. Then σ and τ are homeomorphic.

Proof : Let $v_0, v_1, ..., v_q$ be the vertices of σ, and let $w_0, w_1, ..., w_q$ be the vertices of τ. The required homeomorphism h: $\sigma \to \tau$ is given by

$$h\left(\sum_{j=0}^{q} t_j v_j\right) = \sum_{j=0}^{q} t_j w_j$$

For all $t_0, t_1, ..., t_q$ satisfying $0 \le t_j \le 1$ for $j = 0,1,......,q$ and $\sum_{j=0}^{q} t_j = 1.$

A homeomorphism between two q-simplices defined as in the above proof is referred to as a simplicial homeomorphism.

12.3. SIMPLICIAL COMPLEXES IN EUCLIDEAN SPACES

Definition Let σ and τ be simplices in R^k. We say that τ is a face of σ if the set of vertices of τ is a subset of the set of vertices of σ. A face of σ is said to be a proper face if it is not equal to σ itself. An r-dimensional face of σ is referred to as an r-face of σ. A 1-dimensional face of σ is referred to as an edge of σ.

Note that any simplex is a face of itself. Also the vertices and edges of any simplex are by definition faces of the simplex.

Definition: A finite collection K of simplices in R^k is said to be a simplicial complex if the following two conditions are satisfied:

- if σ is a simplex belonging to K then every face of σ also belongs to K,
- if σ_1 and σ_2 are simplices belonging to K then either $\sigma_1 \cap \sigma_2 = \varnothing$ or else $\sigma_1 \cap \sigma_2$ is a common face of both σ_1 and σ_2.

The dimension of a simplicial complex K is the greatest non-negative integer n with the property that K contains an n-simplex. The union of all the simplices of K is a compact subset $|K|$ of R^k referred to as the polyhedron of K. (The polyhedron is compact since it is both closed and bounded in R^k)

Example: Let K_σ consist of some n-simplex σ together with all of its faces. Then K_σ is a simplicial complex of dimension n, and $|K_\sigma| = \sigma$.

Lemma 12.2 Let K be a simplicial complex, and let X be a topological space. A function f: $f : |K| \to X$ is continuous on the polyhedron $|K|$ of K if and only if the restriction of f to each simplex of K is continuous on that simplex.

Proof: If a topological space can be expressed as a finite union of closed subsets, then a function is continuous on the whole space if and only if its restriction to each of the closed subsets is continuous on that closed set. The required result is a direct application of this general principle.

We shall denote by Vert K the set of vertices of a simplicial complex K (i.e., the set consisting of all vertices of all simplices belonging to K). A collection of vertices of K is said to span a simplex of K if these vertices are the vertices of some simplex belonging to K.

Definition: Let K be a simplicial complex in R^k. A subcomplex of K is a collection L of simplices belonging to K with the following property:

- if σ is a simplex belonging to L then every face of σ also belongs to L. Note that every subcomplex of a simplicial complex K is itself a simplicial complex.

Definition Let $v_0, v_1, ..., v_q$ be the vertices of a q-simplex σ in some Euclidean space R^k. We define the interior of the simplex σ to be the set of all points of σ that are of the form $\sum_{j=0}^{q} t_j v_j$, where $t_j > 0$ for $j = 0,1,.....,q$ and $\sum_{j=0}^{q} t_j = 1$. One can readily verify that the interior of the simplex σ consists of all points of σ that do not belong to any proper face of σ. (Note that, if $\sigma \in R^k$, then the interior of a simplex defined in this fashion will not coincide with the topological interior of σ unless dim σ = k.)

Note that any point of a simplex σ belongs to the interior of a unique face of σ. Indeed let $v_0, v_1, ..., v_q$ be the vertices of σ, and let $x \in \sigma$. Then $x = \sum_{j=0}^{q} t_j v_j$, where $0 \le t_j \le 1$ for $j = 0,1,....,q$ and $\sum_{j=0}^{q} t_j = 1$. The unique face of σ containing x in its interior is then the face spanned by those vertices v_j for which $t_j > 0$.

Lemma 12.3 Let K be a finite collection of simplices in some Euclidean space R^k ,and let $|K|$ be the union of all the simplices in K. Then K is a simplicial complex (with polyhedron $|K|$) if and only if the following two conditions are satisfied:

- K contains the faces of its simplices,
- every point of $|K|$ belongs to the interior of a unique simplex of K.

Proof Suppose that K is a simplicial complex. Then K contains the faces of its simplices. We must show that every point of $|K|$ belongs to the interior of a unique simplex of K. Let $x \in |K|$. Then x belongs to the interior of a face σ of some simplex of K (since every point of a simplex belongs to the interior of some face). But then $\sigma \in K$, since K contains the faces of all its simplices. Thus x belongs to the interior of at least one simplex of K.

Suppose that x were to belong to the interior of two distinct simplices σ and τ of K. Then x would belong to some common face $\sigma \cap \tau$ of σ and τ (since K is a simplicial complex). But this common face would be a proper face of one or other of the simplices σ and τ (since $\sigma \neq \tau$), contradicting the fact that x belongs to the interior of both σ and τ . We conclude that the simplex σ of K containing x in its interior is uniquely determined, as required.

Conversely, we must show that any collection of simplices satisfying the given conditions is a simplicial complex. Since K contains the faces of all its simplices, it only remains to verify that if σ and τ are any two simplices of K with non-empty intersection then $\sigma \cap \tau$ is a common face of σ and τ .

Let $x \in \sigma \cap \tau$. Then x belongs to the interior of a unique simplex ω of K. However any point of σ or τ belongs to the interior of a unique face of that simplex, and all faces of σ and τ belong to K. It follows that ω is a common face of σ and τ , and thus the vertices of ω are vertices of both σ and τ . We deduce that the simplices σ and τ have vertices in common, and that every point of $\sigma \cap \tau$ belongs to the common face ρ of σ and τ spanned by these common vertices. But this implies that $\sigma \cap \tau = \rho$, and thus $\sigma \cap \tau$ is a common face of both σ and τ , as required.

Definition A triangulation (K, h) of a topological space X consists of a simplicial complex K in some Euclidean space, together with a homeomorphism $h : |K| \to K$ mapping the polyhedron $|K|$ of K onto X.

The polyhedron of a simplicial complex is a compact Hausdorff space. Thus if a topological space admits a triangulation then it must itself be a compact Hausdorff space.

Lemma 12.4 Let X be a Hausdorff topological space, let K be a simplicial complex, and let $h:|K|\to X$ be a bijection mapping $|K|$ onto X. Suppose that the restriction of h to each simplex of K is continuous on that simplex. Then the map $h:|K|\to X$ is a homeomorphism, and thus (K, h) is a triangulation of X.

Proof: Each simplex of K is a closed subset of $|K|$, and the number of simplices of K is finite. It follows from Lemma 12.2 that $h:|K|\to X$ is continuous. Also the polyhedron $|K|$ of K is a compact topological space. But every continuous bijection from a compact topological space to a Hausdorff space is a homeomorphism. Thus (K; h) is a triangulation of X.

12.4 SIMPLICIAL MAPS

Definition A simplicial map $\varphi: K\to L$ between simplicial complexes K and L is a function φ: Vert K $\to$ Vert L from the vertex set of K to that of L such that $\varphi(v_0),\varphi(v_1),.......,\varphi(v_q)$ span a simplex belonging to L whenever $v_0,v_1,.....,v_q$ span a simplex of K.

Note that a simplicial map $\varphi: K\to L$ between simplicial complexes K and L can be regarded as a function from K to L: this function sends a simplex σ of K with vertices $v_0,v_1,.....,v_q$ to the simplex $\varphi(\sigma)$ of L spanned by the vertices $\varphi(v_0),\varphi(v_1),.......,\varphi(v_q)$.

A simplicial map $\varphi: K\to L$ also induces in a natural fashion a continuous map $\varphi:|K|\to|L|$ between the polyhedra of K and L, where

$$\varphi\left(\sum_{j=0}^{q} t_j v_j\right)=\sum_{j=0}^{q} t_j\varphi(v_j)$$

Whenever $0\le t_j\le 1$ for $j=0,1,.....,q, \sum_{j=0}^{q} t_j=1$, and $v_0,v_1,.....,v_q$ span a simplex of K. The continuity of this map follows immediately from a straightforward application of Lemma 12.2. Note that the interior of a simplex σ of K is mapped into the interior of the simplex $\varphi(\sigma)$ of L.

There are thus three equivalent ways of describing a simplicial map: as a function between the vertex sets of two simplicial complexes, as a function from one simplicial complex to another, and as a continuous map between the polyhedra of two simplicial complexes. In what follows, we shall describe a simplicial map using the representation that is most appropriate in the given context.

12.5 THE CHAIN GROUPS OF A SIMPLICIAL COMPLEX

Let K be a simplicial complex. For each non-negative integer q, let $\Delta_q(K)$ be the additive group consisting of all formal sums of the form

$$n_1(v_0^1, v_1^1, \ldots\ldots, v_q^1) + n_2(v_0^2, v_1^2, \ldots\ldots, v_q^2) + \ldots\ldots\ldots\ldots + n_s(v_0^8, v_1^8, \ldots\ldots, v_q^8),$$

Where $n_1, n_2, \ldots\ldots, n_s$ are integers and $v_0^r, v_1^r, \ldots\ldots\ldots, v_q^r$ are (not necessarily distinct) vertices of K that span a simplex of K for $r = 1, 2, \ldots\ldots, s.$ (In more formal language, the group $\Delta_q(K)$ is the free Abelian group generated by the set of all $(q+1)$-tuples of the form $(v_0, v_1, \ldots\ldots, v_q)$, where $v_0, v_1, \ldots\ldots, v_q$ span a simplex of K.)

We recall some basic facts concerning permutations. A permutation of a set S is a bijection mapping S onto itself. The set of all permutations of some set S is a group; the group multiplication corresponds to composition of permutations. A transposition is a permutation of a set S which interchanges two elements of S, leaving the remaining elements of the set fixed. If S is finite and has more than one element then any permutation of S can be expressed as a product of transpositions. In particular any permutation of the set $\{0, 1, \ldots\ldots, q\}$ can be expressed as a product of transpositions $(j-1, j)$ that interchange $j-1$ and j for some j.

Associated to any permutation π of a finite set S is a number $\in_\pi$, known as the parity or signature of the permutation, which can take on the values ± 1. If π can be expressed as the product of an even number of transpositions, then $\in_\pi = +1$; if π can be expressed as the product of an odd number of transpositions then $\in_\pi = -1$. The function $\pi \mapsto \in_\pi$ is a homomorphism from the group of permutations of a finite set S to the multiplicative group $\{+1,-1\}$ (i.e., $\in_{\pi\rho} = \in_\pi \in_\rho$ for all permutations π and ρ of the set S). Note in particular that the parity of any transposition is -1.

Definition The q^{th} chain group $C_q(K)$ of the simplicial complex K is defined to be the quotient group $\dfrac{\Delta_q(K)}{\Delta_q^0(K)}$, where $\Delta_q^0(K)$ is the subgroup of $\Delta_q(K)$ generated by elements of the form $(v_0, v_1, \ldots\ldots, v_q)$ where $v_0, v_1, \ldots\ldots, v_q$ are not all distinct, and by elements of the form

$(v_{\pi(0)}, v_{\pi(1)}, \ldots\ldots, v_{\pi(q)}) - \in_\pi (v_0, v_1, \ldots\ldots, v_q)$

Where π is some permutation of $\{0,1,\ldots\ldots,q\}$ with parity $\in_\pi$. For convenience, we define $C_q(K) = \{0\}$ when $q < 0$ or $q >$ dim K, where dim K is the dimension of the simplicial complex K. An element of the chain group $C_q(K)$ is referred to as q-chain of the simplicial complex K. We denote by $\langle v_0, v_1, \ldots\ldots, v_q \rangle$ the element $\Delta_q^0(K) + (v_0, v_1, \ldots\ldots, v_q)$ of $C_q(K)$ corresponding to $(v_0, v_1, \ldots\ldots, v_q)$. The following results follow immediately from the definition of $C_q(K)$.

Lemma 12.5: Let $v_0, v_1, \ldots\ldots, v_q$ be vertices of a simplicial complex K that span a simplex of K. Then

- $\langle v_0, v_1, \ldots\ldots, v_q \rangle = 0$ if $v_0, v_1, \ldots\ldots, v_q$ are not all distinct,
- $\langle v_{\pi(0)}, v_{\pi(1)}, \ldots\ldots, v_{\pi(q)} \rangle = \in_\pi \langle v_0, v_1, \ldots\ldots, v_q \rangle$ for any permutation π of

the set $\{0,1,\ldots\ldots\ldots,q\}$.

Example : If v_0 and v_1 are the endpoints of some line segment then $\langle v_0, v_1 \rangle = -\langle v_1, v_0 \rangle$. If v_0, v_1 and v_2 are the vertices of a triangle in some Euclidean space then

$\langle v_0, v_1, v_2 \rangle = \langle v_1, v_2, v_0 \rangle = \langle v_2, v_0, v_1 \rangle = -\langle v_2, v_1, v_0 \rangle$

$= -\langle v_0, v_2, v_1 \rangle = -\langle v_1, v_0, v_2 \rangle$.

Definition: An oriented q-simplex is an element of the chain group $C_q(K)$ of the form $\pm\langle v_0, v_1, ..., v_q\rangle$, where $v_0, v_1, ..., v_q$ are distinct and span a simplex of K.

An oriented simplex of K can be thought of as consisting of a simplex of K (namely the simplex spanned by the prescribed vertices), together with one of two possible `orientations' on that simplex. Any ordering of the vertices determines an orientation of the simplex; any even permutation of the ordering of the vertices preserves the orientation on the simplex, whereas any odd permutation of this ordering reverses orientation.

Any q-chain of a simplicial complex K can be expressed as a sum of the form

$$n_1\sigma_1 + n_2\sigma_2 + ... + n_s\sigma_s$$

Where $n_1, n_2, ..., n_s$ are integers and $\sigma_1, \sigma_2, ..., \sigma_s$ are oriented q-simplices of K. If we reverse the orientation on one of these simplices σ_i then this reverses the sign of the corresponding coefficient n_i. If $\sigma_1, \sigma_2, ..., \sigma_s$ represent distinct simplices of K then the coefficients $n_1, n_2, ..., n_s$ are uniquely determined.

Example: Let v_0, v_1 and v_2 be the vertices of a triangle in some Euclidean space. Let K be the simplicial complex consisting of this triangle, together with its edges and vertices. Every 0-chain of K can be expressed uniquely in the form

$n_0\langle v_0\rangle + n_1\langle v_1\rangle + n_2\langle v_2\rangle$

for some $n_0, n_1, n_2 \in R$. Similarly any 1-chain of K can be expressed uniquely in the form $m_0\langle v_1, v_2\rangle + m_1\langle v_2, v_0\rangle + m_2\langle v_0, v_1\rangle$

for some $m_0, m_1, m_2 \in R$, and any 2-chain of K can be expressed uniquely as n $\langle v_0, v_1, v_2\rangle$ for some integer n. Lemma 12.6 Let K be a simplicial complex, and let A be an additive group.

Suppose that, to each $(q+1)$-tuple $(v_0, v_1, ..., v_q)$ of vertices spanning a simplex of K, there corresponds an element $\alpha(v_0, v_1, ..., v_q)$ of A, where

Notes

- $\alpha\left(v_0, v_1, \ldots\ldots, v_q\right) = 0$ unless $v_0, v_1, \ldots\ldots, v_q$ are all distinct,
- $\alpha\left(v_0, v_1, \ldots\ldots, v_q\right) = 0$ changes sign on interchanging any two adjacent vertices v_{j-1} and v_j.

Then there exists a well-defined homomorphism from $C_q(k)$ to A which sends $\langle v_0, v_1, \ldots\ldots, v_q\rangle$ to $\left(v_0, v_1, \ldots\ldots, v_q\right)$ whenever $v_0, v_1, \ldots\ldots, v_q$ span a simplex of K. This homomorphism is uniquely determined.

Proof: The given function defined on $(q+1)$-tuples of vertices of K extends to a well-defined homomorphism $\alpha : \Delta_q(K) \to A$ given by

$$\alpha\left(\sum_{r=1}^{s} n_r\left(v_0^r, v_1^r, \ldots\ldots, v_q^r\right)\right) = \sum_{r=1}^{s} n_r \alpha\left(v_0^r, v_1^r, \ldots\ldots, v_q^r\right).$$

for all $\sum_{r=1}^{s} n_r \alpha\left(v_0^r, v_1^r, \ldots\ldots, v_q^r\right) \in \Delta_q(K)$. Moreover $\left(v_0, v_1, \ldots\ldots, v_q\right) \in$ ker α unless $v_0, v_1, \ldots\ldots, v_q$ are all distinct. Also

$$\left(v_{\pi(0)}, v_{\pi(1)}, \ldots\ldots, v_{\pi(q)}\right) - \in_\pi \left(v_0, v_1, \ldots\ldots, v_q\right) \in \ker \alpha$$

For all permutations π of $\{0, 1, \ldots\ldots, q\}$ since the permutation π can be expressed as a product of transpositions $(j-1, j)$ that interchange $j-1$ with j for some j and leave the rest of the set fixed, and the parity $\in_\pi$ of π is given by $\in_\pi = +1$ when the number of such transpositions is even, and by $\in_\pi = -1$ when the number of such transpositions is odd. Thus the generators of $\Delta_q^0(K)$ are contained in ker α, and hence $\Delta_q^0(K) \subset$ ker α. The required homomorphism $\alpha C_q(K) \to A$ is then defined by the formula

$$\alpha\left(\sum_{r=1}^{s} n_r \left\langle v_0^r, v_1^r, \ldots\ldots, v_q^r\right\rangle\right) = \sum_{r=1}^{s} n_r \alpha\left(v_0^r, v_1^r, \ldots\ldots, v_q^r\right).$$

12.6 BOUNDARY HOMOMORPHISMS

Let K be a simplicial complex. We introduce below boundary homomorphisms $\partial_q : C_q(K) \to C_{q-1}(K)$ between the chain groups of K. If σ is an oriented q-simplex of K then $\partial_q(\sigma)$ is a $(q-1)$-chain which is a formal sum of the $(q-1)$-faces of σ, each with an orientation determined by the orientation of σ.

Let σ be a q-simplex with vertices $v_0, v_1, ..., v_q$. For each integer j between 0 and q we denote by $\langle v_0, ..., v_{j-1}, ..., v_q \rangle$ the oriented $(q-1)$-face $\langle v_0, ..., v_{j-1}, v_{j+1}, ..., v_q \rangle$

of the simplex σ obtained on omitting v_j from the set of vertices of σ.

In particular

$$\langle \hat{v}_0, v_1, ..., v_q \rangle \equiv \langle v_1, ..., v_q \rangle, \qquad \langle v_0, ..., v_{q-1}, \hat{v}_q \rangle \equiv \langle v_0, ..., v_{q-1} \rangle.$$

Similarly if j and k are integers between 0 and q, where $j < k$, we denote by $\langle v_0, ..., \hat{v}_j, ..., \hat{v}_k, ...v_q \rangle$

the oriented $(q-2)$-face $\langle v_0, ..., v_{j-1}, v_{j+1}, ..., v_{k-1}, v_{k+1}, ..., v_q \rangle$ of the simplex σ obtained on omitting v_j and v_k from the set of vertices of σ We now define a `boundary homomorphism' $\partial_q : C_q(K) \to C_{q-1}(K)$ for each integer q. Define $\partial_q = 0$ if $q \le 0$ or $q >$ dim K. (In this case one or other of the groups $C_q(K)$ and $C_{q-1}(K)$ is trivial.) Suppose then that $0 < q \le$ dim K. Given vertices $\alpha(v_0, v_1, ..., v_q)$ spanning a simplex of K, let

$$\alpha(v_0, v_1, ..., v_q) = \sum_{j=0}^{q} (-1)^j \langle v_0, ..., \hat{v}_1, ..., v_q \rangle$$

Inspection of this formula shows that $\alpha(v_0, v_1, ..., v_q)$ changes sign whenever two adjacent vertices v_{i-1} and v_i are interchanged.

Suppose that $v_j = v_k$ for some j and k satisfying $j < k$. Then

$$\alpha(v_0, v_1 ...., v_q) = (-1)^j \langle v_0, ...\hat{v}_j, ..., v_q \rangle + (-1)^k \langle v_0, ...\hat{v}_k, ..., v_q \rangle,$$

since the remaining terms in the expression defining $\alpha(v_0, v_1 ...., v_q)$ contain both v_j and v_k. However $(v_0, ...\hat{v}_k, ..., v_q)$ can be transformed to $(v_0, ...\hat{v}_j, ..., v_q)$ by making $k - j - 1$ transpositions which interchange v_j successively with the vertices $v_{j+1}, v_{j+2}, ..., v_{k-1}$.

Therefore

$$\langle v_0,.......\hat{v}_k,.......,v_q\rangle = (-1)^{k-j-1}\langle v_0,.......\hat{v}_j,.......,v_q\rangle$$

Thus $\alpha(v_0,v_1,......,v_q)=0$ unless $v_0,v_1,......,v_q$ are all distinct. It now follows immediately from Lemma 12.2 that there is a well-defined homomorphism $\partial_q : C_q(K)\to C_{q-1}(K)$, characterized by the property that

$$\partial_q\left(\langle v_0,v_1,......,v_q\rangle\right)=\sum_{j=0}^{q}(-1)^j\langle v_0,......,\hat{v}_1,......,v_q\rangle$$

Whenever $v_0,v_1,......,v_q$ span a simplex of K. Lemma 12.7: $\partial_{q-1}\ o\ \partial_q = 0$ for all integers q.

Proof: The result is trivial if $q<2$, since in this case $\partial_{q-1}=0$. Suppose that $q\geq 2$. Let $v_0,v_1,....,v_q$ be vertices spanning a simplex of K. Then

$$\partial_{q-1}\partial_q\left(\langle v_0,v_1,......,v_q\rangle\right)=\sum_{j=0}^{q}(-1)^j\,\partial_{q-1}\left(\langle v_0,......,\hat{v}_j,......,v_q\rangle\right)$$

$$=\sum_{j=0}^{q}\ \sum_{k=0}^{j-1}(-1)^{j+k}\langle v_0,......,\hat{v}_k,......,\hat{v}_j,......,v_q\rangle$$

$$+\sum_{j=0}^{q}\ \sum_{k=j+1}^{q}(-1)^{j+k-1}\langle v_0,......,\hat{v}_j,......,\hat{v}_k,......,v_q\rangle$$

$$=0$$

(Since each term in this summation over j and k cancels with the corresponding term with j and k interchanged). The result now follows from the fact that the homomorphism $\partial_{q-1}\circ\partial_q$ is determined by its values on all oriented q-simplices of K.

12.7 THE HOMOLOGY GROUPS OF A SIMPLICIAL COMPLEX

Let K be a simplicial complex. A q-chain z is said to be a q-cycle if $\partial_q z = 0$. A q-chain b is said to be a q-boundary if $b=\partial_{q+1}c'$ for some $(q+1)$-chain c'. The group of q-cycles of K is denoted by $Z_q(K)$, and the group of q-boundaries of K is denoted by $B_q(K)$. Thus $Z_q(K)$ is the kernel of the boundary homomorphism $\partial_q : C_q(K) \to C_{q-1}(K)$, and $B_q(K)$ is the image of the boundary homomorphism $\partial_{q+1} : C_{q+1}(K) \to C_q(K)$. However $\partial_q \circ \partial_{q+1} = 0$, by Lemma 12.3. Therefore $B_q(K) \subset Z_q(K)$. But these groups are subgroups of the Abelian group $C_q(K)$. We can therefore form the quotient group $H_q(K)$, where $\frac{H_q(K) = Z_q(K)}{B_q(K)}$. The group $H_q(K)$ is referred to as the qth homology group of the simplicial complex K. Note that $H_q(K) = 0$ if $q < 0$.

or $q >$ dim K (since $Z_q(K) = 0$ and $B_q(K) = 0$ in these cases). It can be shown that the homology groups of a simplicial complex are topological invariants of the polyhedron of that complex.

The element $[z] \in H_q(K)$ of the homology group $H_q(K)$ determined by $z \in Z_q(K)$ is referred to as the homology class of the q-cycle z. Note that $[z_1 + z_2] = [z_1] + [z_2]$ for all $z_1 z_2 \in Z_q(K)$, and $[z_1] = [z_2]$ if and only if $z_1 - z_2 = \partial_{q+1} c$ for some $(q+1)$-chain c.

Proposition 12.8: Let K be a simplicial complex. Suppose that there exists a vertex w of K with the following property:

- If vertices $v_0, v_1, \ldots\ldots, v_q$ span a simplex of K then so do $w, v_0, v_1, \ldots\ldots, v_q$.

Then $H_0(K) \cong R$, and $H_q(K)$ is the zero group for all $q > 0$.

Proof: Using Lemma 12.6, we see that there is a well-defined homomorphism $D_q : C_q(K) \to C_{q+1}(K)$ characterized by the property that $D_q(\langle v_0, v_1, \ldots\ldots, v_q \rangle) = \langle w, v_0, v_1, \ldots\ldots, v_q \rangle$

whenever $v_0, v_1, \ldots\ldots, v_q$ span a simplex of K. Now $\partial_1(D_0(v)) = v - w$ for all vertices v of K. It follows that

$$\sum_{r=1}^{8} n_r \langle v_r \rangle - \left(\sum_{r=1}^{8} n_r \right) \langle w \rangle = \sum_{r=1}^{8} n_r \left(\langle v_r \rangle - \langle w \rangle \right) \in B_0(K)$$

for all $\sum_{r=1}^{8} n_r \langle v_r \rangle \in C_0(K)$. But $Z_0(K) = C_0(K)$ (since $\partial_0 = 0$ by definition), and thus $\frac{H_0(K) = C_0(K)}{B_0(K)}$. It follows that there is a well-defined surjective homomorphism from $H_0(K)$ to R induced by the homomorphism from $C_0(K)$ to R that sends $\sum_{r=1}^{8} n_r \langle v_r \rangle \in C_0(K)$ to $\sum_{r=1}^{8} n_r$. Moreover this induced homomorphism is an isomorphism from $H_0(K)$ to R.

Now let $q > 0$. Then

$$\partial_{q+1}\left(D_q\left(\langle v_0, v_1, ...v_q \rangle\right)\right)$$

$$= \partial_{q+1}\left(\langle w, v_0, v_1, ...v_q \rangle\right)$$

$$= \langle v_0, v_1, ...v_q \rangle + \sum_{j=0}^{q} (-1)^{j+1} \langle w, v_0, ...\hat{v}_1, ..., v_q \rangle$$

$$= \langle v_0, v_1, ...v_q \rangle - D_{q-1}\left(\partial_q\left(\langle v_0, v_1, ..., v_q \rangle\right)\right)$$

Whenever $v_0, v_1, ..., v_q$ span a simplex of K. Thus

$$\partial_{q+1}\left(D_q(c)\right) + D_{q-1}\left(\partial_q(c)\right) = c$$

For all $c \in C_q(K)$. In particular $z = \partial_{q+1}\left(D_q(z)\right)$ for all $z \in Z_q(K)$, and hence Z $Z_q(K) = B_q(K)$. It follows that $H_q(K)$ is the zero group for all $q > 0$, as required.

Example: The hypotheses of the proposition are satisfied for the complex K_σ consisting of a simplex σ together with all of its faces: we can choose w to be any vertex of the simplex σ.

12.8 SIMPLICIAL MAPS AND INDUCED HOMOMORPHISMS

Any simplicial map $\varphi : K \to L$ between simplicial complexes K and L induces well-defined homomorphisms $\varphi_q : C_q(K) \to C_q(L)$ of chain groups, where $\varphi_q(\langle v_0, v_1, ..., v_q \rangle) = \langle \varphi(v_0), \varphi(v_1), ...\varphi(v_q) \rangle$ whenever $v_0, v_1, ..., v_q$ span a simplex of K. (The existence of these induced homomorphisms follows from a straightforward application of Lemma 12.2.) Note that $\varphi_q(\langle v_0, v_1, ..., v_q \rangle) = 0$ unless $\varphi(v_0), \varphi(v_1), ...\varphi(v_q)$ are all distinct. Now $\varphi_{q-1} \circ \partial_q = \partial_q \circ \varphi_q$ for each integer q. Therefore $\varphi_q(Z_q(K)) \subset Z_q(L)$ and $\varphi_q(B_q(K)) \subset B_q(L)$ for all integers q. It follows that any simplicial map $\varphi : K \to L$ induces well-defined homomorphisms $\varphi_* : H_q(K) \to H_q(L)$ of homology groups, where $\varphi_* : ([z]) = [\varphi_q(z)]$ for all q-cycles $z \in Z_q(K)$. It is a trivial exercise to verify that if K, L and M are simplicial complexes and if $\varphi : K \to L$ and $\psi : L \to M$ are simplicial maps then the induced homomorphisms of homology groups satisfy $(\psi \circ \psi)_* = \psi_* \circ \varphi_*$.

12.9 CONNECTEDNESS AND $H_0(K)$

Lemma 12.9 Let K be a simplicial complex. Then K can be partitioned into pairwise disjoint subcomplexes $K_1, K_2 ...., K_r$ whose polyhedral are the connected components of the polyhedron $|K|$ of K.

Proof Let $X_1, X_2, ..., X_r$ be the connected components of the polyhedron of K, and, for each j, let K_j be the collection of all simplices σ of K for which $\sigma \subset X_j$. If a simplex belongs to K_j for all j then so do all its faces. Therefore $K_1, K_2 ...., K_r$ are subcomplexes of K. These subcomplexes are pairwise disjoint since the connected components $X_1, X_2, ..., X_r$ of $|K|$ are

pairwise disjoint. Moreover, if $\sigma \in K$ then $\sigma \subset X_j$ for some j, since σ is a connected subset of $|K|$, and any connected subset of a topological space is contained in some connected component. But then $\sigma \in K_j$. It follows that $K = K_1 \cup K_2 \cup ...... \cup K_r$ and

Notes

$|K| = |K_1| \cup |K_2| \cup ...... \cup |K_r|$, as required.

The direct sum $A_1 \oplus A_2 \oplus ...... \oplus A_r$ of additive Abelian groups $A_1, A_2, ..., A_r$ is defined to be the additive group consisting of all r-tuples $(a_1, a_2, ..., a_r)$ with $a_i \in A_i$ for $i = 1, 2, ...r,$ where

$(a_1, a_2, ..., a_r) + (b_1, b_2, ..., b_r) \equiv (a_1 + b_1, a_2 + b_2, ..., a_r + b_r)$

Lemma 12.10 : Let K be a simplicial complex. Suppose that $K = K_1 \cup K_2 \cup .... \cup K_r$ where $K_1, K_2, ...K_r$ are pairwise disjoint. Then $H_q(K) \cong H_q(K_1) \oplus H_q(K_2) \oplus .... \oplus H_q(K_r)$

for all integers q.

Proof: We may restrict our attention to the case when $0 \le q \le$ dim K, since $H_q(K) = \{0\}$ if $q < 0$ or $q >$ dim K. Now any q-chain c of K can be expressed uniquely as a sum of the form $c = c_1 + c_2 + ... + c_r,$ where c_j is a q-chain of K_j for $j = 1, 2, ..., r.$ It follows that

$C_q(K) \cong C_q(K_1) \oplus C_q(K_2) \oplus .... \oplus C_q(K_r).$

Now let z be a q-cycle of K (i.e., $z \in C_q(K)$ satisfies $\partial_q(z) = 0$). We can express z uniquely in the form $z = z_1 + z_2 + ... + z_r,$ where z_j is a q-chain of K_j for $j = 1, 2, ..., r.$ Now

$0 = \partial_q(z) = \partial_q(z_1) + \partial_q(z_2) + ... + \partial_q(z_r),$

and $\partial_q(z_r)$ is a $(q-1)$-chain of K_j for $j = 1, 2, ..., r.$ It follows that $\partial_q(z_r) = 0 \quad j = 1, 2, ..., r.$ Hence each z_j is a q-cycle of K_j , and thus

$Z_q(K) \cong Z_q(K_1) \oplus Z_q(K_2) \oplus .... \oplus Z_q(K_r).$

Now let b be a q-boundary of K. Then $b = \partial_{q+1}(c)$ for some $(q+1)-$ chain c of K. Moreover $c = c_1 + c_2 + ... + c_r,$ where $c_j \in C_{q+1}(K_j).$ Thus $b_1 + b_2 + ...b_r,$ where $b_j \in B_q(K_j)$ is given by $b_j = \partial_{q+1} c_j$ for $j = 1, 2, ..., r..$ We deduce that

$B_q(K) \cong B_q(K_1) \oplus B_q(K_2) \oplus .... \oplus B_q(K_r).$

It follows from these observations that there is a well-defined isomorphism

$$v : H_q(K_1) \oplus H_q(K_2) \oplus .... \oplus H_q(K_r) \to H_q(K)$$

which maps $([z_1],[z_2],.........,[z_r])$ to $[z_1 + z_2 + ... + z_r]$, where $[z_j]$ denotes the homology class of a q-cycle z_j of K_j for $j = 1, 2, ..., r.$

Let K be a simplicial complex, and let y and z be vertices of K. We say that y and z can be joined by an edge path if there exists a sequence $v_0, v_1, ...v_m$ of vertices of K with $v_0 = y$ and $v_m = z$ such that the line segment with endpoints v_{j-1} and v_j is an edge belonging to K for $j = 1, 2, ..., m.$

Lemma 12.11 : The polyhedron $|K|$ of a simplicial complex K is a connected topological space if and only if any two vertices of K can be joined by an edge path.

Proof: It is easy to verify that if any two vertices of K can be joined by an edge path then $|K|$ is path-connected and is thus connected. (Indeed any two points of $|K|$ can be joined by a path made up of a finite number of straight line segments.) We must show that if $|K|$ is connected then any two vertices of K can be joined by an edge path. Choose a vertex v_0 of K. It suffices to verify that every vertex of K can be joined to v_0 by an edge path Let K_0 be the collection of all of the simplices of K having the property that one (and hence all) of the vertices of that simplex can be joined to v_0 by an edge path. If σ is a simplex belonging to K_0 then every vertex of σ can be joined to v_0 by an edge path, and therefore every face of σ belongs to K_0. Thus K_0 is a subcomplex of K. Clearly the collection K_1 of all simplices of K which do not belong to K_0 is also a subcomplex of K. Thus $K = K_0 \cup K_1$, where $K_0 \cap K_1 = \theta$, and hence $|K| = |K_0| \cup |K_1|$, where $|K_0| \cap |K_1| = \theta$. But the polyhedra $|K_0|$ and $|K_1|$ of K_0 and K_1 are closed subsets of $|K|$. It follows from the connectedness of $|K|$ that either $|K_0| = \theta$ or $|K_1| = \theta$. But $v_0 \in K_0$. Thus $K_1 = \theta$ and $K_0 = K$, showing that every vertex of K can be joined to v_0 by an edge path, as required. Theorem 12.11: Let K be a simplicial complex.

Suppose that the polyhedron $|K|$ of K is connected. Then $H_0(K) \cong R.$

Proof Let $u_1, u_2, ...u_r$ be the vertices of the simplicial complex K. Every 0-chain of K can be expressed uniquely as a formal sum of the form $n_1\langle u_1\rangle + n_2\langle u_2\rangle + ... + n_r\langle u_r\rangle$ for some integers $n_1, n_2, ...n_r$. It follows that there is a well-defined homomorphism $\varepsilon : C_0(K) \to R$ defined by

$\varepsilon\left(n_1\langle u_1\rangle + n_2\langle u_2\rangle + ... + n_r\langle u_r\rangle\right) = n_1, n_2, ...n_r.$

Now $\varepsilon\left(\partial_1(\langle y, z\rangle)\right) = \varepsilon(\langle z\rangle - \langle y\rangle) = 0$ whenever y and z are endpoints of an edge of K. It follows that $\varepsilon \circ \partial_1 = 0,$ and hence B0(K) $B_0(K) \subset$ ker ε .

Let $v_0, v_1, ..., v_m$ be vertices of K determining an edge path. Then

$$\langle v_m\rangle - \langle v_0\rangle = \partial_1\left(\sum_{j=1}^{m}\langle v_{j-1}, v_j\rangle\right) \in B_0(k)$$

Now $|K|$ is connected, and therefore any pair of vertices of K can be joined by an edge path (Lemma 12.7). We deduce that $\langle z\rangle - \langle y\rangle \in B_0(K)$ for all vertices y and z of K. Thus if $c \in$ ker ε, where $c = \sum_{j=1}^{r} n_j\langle u_j\rangle,$ then $\sum_{j=1}^{r} n_j = 0,$ and hence $c = \sum_{j=1}^{r} n_j\left(\langle u_j\rangle - \langle u_1\rangle\right).$

But $\langle u_j\rangle - \langle u_1\rangle \in B_0(K)$. It follows that $c \in B_0(K)$. We conclude that ker $\varepsilon \subset B_0(K)$, and hence ker $\varepsilon = B_0(K)$.

Now the homomorphism $\varepsilon : C_0(K) \to R$ is surjective and its kernel is $B_0(K)$. Therefore, it induces an isomorphism from $\frac{C_0(K)}{B_0(K)}$ to R .

However $Z_0(K) = C_0(K)$ (since $\partial_0 = 0$ by definition). Thus $H_0(K) \equiv C_0(K)/B_0(K)R,$ as required.

On combining Theorem 12.11 with Lemmas 12.9 and 12.10 we obtain immediately the following result. Corollary 12.12 Let K be a simplicial complex. Then $H_0(K) \cong \mathbf{Z} \oplus \mathbf{Z} \oplus .... \oplus \mathbf{Z}$ (r times),

where r is the number of connected components of $|K|$.

Check your progress:

1. Prove: Let q be a non-negative integer, let σ be a q-simplex in R^m, and let τ be a q-simplex in R^n, where $m \geq q$ and $n \geq q$. Then σ and τ are homeomorphic.

2. Prove: Let K be a simplicial complex, and let X be a topological space. A function f: $f : |K| \to X$ is continuous on the polyhedron $|K|$ of K if and only if the restriction of f to each simplex of K is continuous on that simplex.

3. Prove: Let K be a finite collection of simplices in some Euclidean space R^k ,and let $|K|$ be the union of all the simplices in K. Then K is a simplicial complex (with polyhedron $|K|$) if and only if the following two conditions are satisfied: K contains the faces of its simplices, every point of $|K|$ belongs to the interior of a unique simplex of K.

4. Prove: Let K be a simplicial complex. Suppose that there exists a vertex w of K with the following property: If vertices $v_0, v_1, \ldots\ldots, v_q$ span a simplex of K then so do $w, v_0, v_1, \ldots\ldots, v_q$. Then $H_0(K) \cong R,$ and $H_q(K)$ is the zero group for all $q > 0$.

12.10 LET US SUM UP

1. Points $v_0, v_1, \ldots\ldots, v_q$ in some Euclidean space R^k are said to be

geometrically independent (or affine independent) if the only solution of the linear system

$$\begin{cases} \sum_{j=0}^{q} \lambda_j v_j = 0 \\ \sum_{j=0}^{q} \lambda_j = 0 \end{cases}$$

is the trivial solution $\lambda_0 = \lambda_1 = ... = \lambda_q = 0.$

2. A q-simplex in R^k is defined to be a set of the form

$$\left\{ \sum_{j=0}^{q} t_j v_j : 0 \le t_j \le 1 \ for \ \ j = 0,1,....q \ and \sum_{j=0}^{q} t_j = 1 \right\},$$

where $v_0, v_1, ..., v_q$ are geometrically independent points of R^k. The points $v_0, v_1, ..., v_q$ are referred to as the vertices of the simplex. The non-negative integer q is referred to as the dimension of the simplex.

3. Let K be a simplicial complex in R^k. A subcomplex of K is a collection L of simplices belonging to K with the following property: If σ is a simplex belonging to L then every face of σ also belongs to L. Note that every subcomplex of a simplicial complex K is itself a simplicial complex.

4. Let K be a finite collection of simplices in some Euclidean space R^k ,and let $|K|$ be the union of all the simplices in K. Then K is a simplicial complex (with polyhedron $|K|$) if and only if the following two conditions are satisfied:

- K contains the faces of its simplices, every point of $|K|$ belongs to the interior of a unique simplex of K.

5. An oriented q-simplex is an element of the chain group $C_q(K)$ of the form $\pm\langle v_0, v_1, ..., v_q \rangle$, where $v_0, v_1, ..., v_q$ are distinct and span a simplex of K.

12.11 KEY WORDS

Geometrical Independence

Euclidean spaces

Simplical complex

Boundary Homomarphism

Homology groups

Simplical maps

12.12 QUESTIONS FOR REVIEW

1. Explain about The chain groups of simplical complex.
2. Explain about boundary homomorphism.
3. Explain about simplical maps and induced homomorphism

12.13 SUGGESTIVE READINGS AND REFERENCES

1. Algebraic Topology – Satya Deo

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12.14 ANSWERS TO CHECK YOUR PROGRESS

1. See section 12.3
2. See section 12.4
3. See section 12. 4
4. See section 12.8

UNIT-13 ALGEBRAIC CURVES I

STRUCTURE

13.0 Objective

13.1 Introduction

13.2 The Kodaira embedding

13.3 Riemann- Roch theorem

13.4 General results on Algebraic curves

13.5 Let us sum up

13.6 Key words

13.7 Questions for review

13.8 Suggestive readings and references

13.9 Answers to check your progress

13.0 OBJECTIVE

In this unit we will learn and understand about the Kodaira embedding, Riemann- Rock theorem and general results on Algebraic Curves.

13.1 INTRODUCTION

The main purpose of this chapter is to show that compact Riemann surfaces can be imbedded into projective space (i.e. they are algebraic curves), and to study some of their basic properties.

13.2 THE KODAIRA EMBEDDING

We start by showing that any compact Riemann surface can be embedded as a smooth subvariety in projective space P_N : this is special instance of the so-called Kodiara's embedding theorem. Together with Chow's Lemma this implies that every compact Riemann surface is algebraic.

We recall that, given two complex manifolds $X \, and \, Y$, we say that (Y,t) is a sub manifold of X is ι is an injective holomorphic map $Y \to X$ whose differential $\iota_{*P} : T_P Y \to T_{i(p)} X$ is of maximal rank (given by the dimension of Y)at all $p \in Y$. In other terms, ι maps iso morphically Y onto a smooth sub variety of X .

PROPOSITION 13.1. Any compact Riemann surface can be realized as a sub manifold of P_N for some N.

Proof. Pick up a line bundle L on S such that deg $L > \deg K + 2$ (choose an effective divisor D with enough points, and let $L = [D]$). By Serre duality we have

(13.1) $H^{'}(S, O(L-2p)) \square\, H^0(S, O(L-2p)^{-1} \otimes K)^* = 0$

For any $p \in S$, since deg $(K - L + 2p) < 0 (here L - 2p = L \otimes [-2o])$. Consider now the exact sequence

$$0 \to O(L-2p) \to O(L) \xrightarrow{d_p \otimes ev_p} T_p^* S \otimes L_p \to 0$$

(the morphism d_p is Cartan's differential followed by evaluation at P ,while ev_p is the evaluation of sections at p).Due to (8.1) we get

$$0 \to H^0(S, O(L-2p)) \to H^0(S, O(L)) \xrightarrow{d_p \otimes ev_p} T_P^* S \oplus L_p \to 0$$

So that dim $|D| \geq 1$. Let $N = \dim |D|, and \, let \{so, ..., s_N\}$ be a basis of $|D|$. If U is an open neighbourhood of $p, and \, \phi L_{/U} \to U \times \mathbf{C}$ is a local trivialization of L, the quantity

$$[\phi \circ s_0, ..., \phi \circ S_N] \in \mathrm{P}_N$$

Does not depend on the trivialization ϕ ; we have therefore established a (holomorphic) map $\iota_L : S \to \mathrm{P}_{N.}^{\ 1}$ We must prove that (1) ι_L is injective, and (2) the differential $(\iota_L)_*$ never vanishes. (1) It is enough to prove that, given any two points $p,q \in S$, there is a section $s \in H^0(S,\mathrm{O}(L))$ such that $s(p) \neq \lambda s(q)$ for all $\lambda \in C^*$; this in turn implied by the surjectivity of the map

$$H^0(S,O(L)) \xrightarrow{r_{p,q}} L_p \oplus L_q, \qquad s \mapsto s(p)+s(q).$$

To show this we start from the exact sequence

$$0 \to O(L-p-q) \to O(L) \xrightarrow{r_{p,q}} L_p \oplus L_q \to 0$$

and note that in coholomology we have

$$H^0(S,O(L-p-q)) \xrightarrow{r_{p,q}} L_p \oplus L_q \to H^1(S,O(L-p-q)) = 0$$

Since

$$H^1(S,O(L-p-q)) \cong H^0(S,O(l-p-q)^{-1} \otimes K)^* = 0$$

Because deg $(L-p-q)^{-1} \otimes K = \deg K - \deg L + 2 < 0.$

(2) We shall actually show that the adjoint map $(\iota L)^* : T^*_{\iota L(p)}\mathrm{P}_N \to T^*_p S$ is surjective. The cotangent space $T^*_p S$ can be realized as the space of equivalence classes of holomorphic functions which have the same value at p (e.g; the zero value) and have a first-0rder contact (i.e., they have the same differential at p). Let ϕ be a trivializing map for L around p; we must find a section $s \in H^0(S,O(L))$ such that $\phi \circ s(p) = 0 (i.e. s(p)=0) and\, (\phi \circ s)^*$ is surjective at p. This is equivalent to showing that the map $H^0(S,O(L-p)) \xrightarrow{d_p} T^*_P$ is surjective, since $O(L-p)$ is the sheaf of holomorphic sections of L vanishing at p. We consider the exact sheaf sequence

$$0 \to O(L-2p) \to O(L-p) \xrightarrow{d_p} T^*_p S \to 0;$$

By Serre duality,

$$H^1(S,O(L-2p))^* \cong H^0(S,O(-L+2p+K)) = 0$$

so that $H^0(S, O(L-p)) \xrightarrow{d_p} T_p^* S$ is surjective.

Given any complex manifold X, one says that a line bundle $L\, on\, X$ is very ample if the construction (13.2) defines an imbedding of X into $\mathrm{P}H^0(X, O(L))$. A line bundle L is said to be ample if L^n is very ample for some natural n. A sufficient condition for a line bundle to be ample may be stated as follows. Definition 13.2. A (1,1) form ω on a complex manifold is said to be positive if it can be locally written in the form

$$\omega = i\omega_{ij} dz^i \wedge d_z^{-j}$$

This map actually depends on the choice of a basis of $|D|$; however, different choices correspond to an action of the group $\mathrm{P}Gl(N+1, \mathrm{C})\mathrm{P}_N$ and therefore produce isomorphic subvarieties of P_N with ω_{ij} a positive definite hermitian matrix.

DEFINITION 13.3. If the first Chern class of a line bundle L on a complex manifold can be represented by a positive 2-form, then L is ample.

While we have seen that any compact Riemann surface carries plenty of very ample line bundles, this in general is not the case: there are indeed complex manifolds which cannot be imbedded into any projective space.

A first consequence of the imbedding theorem expressed by Proposition 8.1 is that any line bundle on a compact Riemann surface comes from a divisor, i.e. Div (S)/linear equivalence $\cong Pie(S)$.

PROPOSITION 13.4. If M is a smooth 1-dimensional 2 analytic submanifold of projective space P_n (i.e. M is the imbedding of a compact Riemann surface into P_n),and L is a line bundle on M, there is a divisor $D\, on\, M$ such that $L = [D]$.

Proof. We must find a global meromorphic section of L .Let H_M be the restriction to M of the hyperplane bundle $H\, of\, \mathrm{P}_n$, and let V be the intersection of M with a hyperplane in P_n (so $[V] \square H_M$, and since V is effective, H_M has global holomorphic sections).We shall show that for a big enough integer m the line bundle $L+mH_M$ $(= L+\otimes H_M^m)$ has a global holomorphic section $s; if\ t$ is a holomorphic section of H_M, the required meromorphic section of $L\, is\, s/t^m$.

We have an exact sequence

$$0 \to O_M(-H_M) \xrightarrow{s} O_M \to kv \to 0$$

So that after tensoring by $L+mH_M$,

13.3 $0 \to O_M(L+(m-1)H_M) \xrightarrow{s} O_M(L+mH_M) \to kv \to 0.$

(Here $\xrightarrow{s}$ denotes the morphism given by multiplication by s). The associated long cohomology exact sequence contains the segment

$H^0(M, O_M(L+mH_M)) \xrightarrow{s} \mathrm{C}^{\mathrm{N}} \to H^1(M.O_M(L+(m-1)H_M))$

Where $N = \deg^V .But$

$$H^1(M, O_M(L+(m-1)H_M)) \square H^0(M, K_M \otimes \circ(L-(m-1)H_M))^* = 0$$

By Serre duality and the vanishing theorem (if m is big enough, deg $K_M \otimes O(-L-(m-1)H_M) < 0).$ Therefore the morphism r in (8.3) surpjective, and $H^0(M, O_M(L+mH_M)) \neq 0.$

We shall now proceed to identify compact Riemann surfaces with (smooth) algebraic curves. Given a homogeneous polynomial $F\, on\, C^{n+1}$ the zero locus of $F\, in\, C^{n+1}$.

DEFINITION 13.5. A (projective) algebraic variety is a subvariety of P_n which is the zero locus of finite collection of homogeneous polynomials. We shall say that an algebraic variety is smooth if it is so as an analytic subvariety of P_n

The dimension of an algebraic variety is its dimension as an analyitic subvariety of P_n. A one-dimensional algebraic variety is called an algebraic curve.

The following fundamental result, called Chows lemma, it is not hard to prove; we shall anyway omit its proof for the sake of brevity.

PROPOSITION 13.6. (Chow's lemma) Any analytic subvariety of P_n is algebraic.

Exercise13.7. Use Chow's lemma to show that $H^0(\mathrm{P}_n, H^d)$ -where H is the hyperplane line bundle- can be identified with the space of homogenous polynomials of degree d on C^{n+1}.

Using Chow's lemma together with the imbedding theorem (Proposition 13.1) we obtain

COROLLARY 13.8. Any compact Ricemann surface is smooth algebraic curve.

We switch from the terminology "compact Riemann surface" to "algebraic curve",

understanding that we shall only consider smooth algebraic curves. [3]

We shall usually denote an algebraic curve by the letter C.

1. Riemann-Roch theorem
2. A fundamental result in the study of algebraic curves

We switch from the terminology "compact Riemann surface" to "algebraic curve", understanding that we shall only consider smooth algebraic curves".

We shall usually denote an algebraic curve by the letter C.

13.3 RIEMANN-ROCH THEOREM

A fundamental result in the study of algebraic curves in the Riemann-Roch theorem. Let C be an algebraic curve, and denote by K its canonical bundle. 4 We denote $g = h^0(K)$, and call it the arithmetic genus of C (this number will be shortly identified with the topological genus of C).

Proposition 8.1. (Riemann-Roch theorem) For any line bundle L on C one has

$$h^o(L) - h^1(L) = \deg L - g + 1.$$

Proof. If $L = \mathrm{C}$ is the trivial line bundle, the result holds obviously (notice that $H^1(C,O)^* \cong H^0(C,K)$ by Serre duality).Exploiting the fact that $L = [D]$ for some divisor D, it is enough to prove that if the results hold for $L = [D]$, then it also holds for $L^{'} = [D+p] \, and \, L" = [D-p]$.

In the first case we start from the exact sequence

$0 \to O(D) \to (D+p) \to k_p \to 0$

3 Strictly speaking an algebraic curve consists of more data than a compact Riemann surface S, since the former requires an imbedding of S into a projective space, ie. The choice of an ample line bundle.

4 We introduce the following notation if ε is a sheaf of Oc -modules, then $h^i(\varepsilon) = \dim H^i(C,\varepsilon)$.

13.4 GENERAL RESULTS ABOUT ALGEBRAIC CURVES

Which gives (since $H^1(C,k_p) = 0$)

$$0 \to H^0(S,O(D)) \to H^0(S,O(D+p)) \to C \to H^1(S,O(D)) \to H^1(S,O(D+p)) \to 0$$

Hence

$$h^0(^{'}) - h^0(L) - h^1(L) + 1 = \deg L - g + 2 = \deg L^{'} - g + 1.$$

Analogously for $h^0(^{'}) - h^0(L) - h^1(L) + 1 = \deg L - g + 2 = \deg L^{'} - g + 1.$

Analogously for $L^{''}$.

By using the Riemann-Roch theorem and Serre duality we may compute the degree of K, obtaining

$$\deg K = 2g - 2.$$

This is called the Riemann-Hurwitz formula. It allows us to identify g with the topological genus g_{top} of C regarded as a compact oriented 2-

dimensional real manifold S. To this end we may use the Gauss-Bonnet theorem, which states that the integral of the Euler class of the real tangent bundle to S is the Euler characteristic of $S, \chi(S) = 2 - 2_{gtop.}$ On the other hand the complex structure of C makes the real tangent bundle into a complex holomorphic line bundle, isomorphic to the holomorphic tangent bundle TC, and under this identification the Euler class corresponds to the first Chern class of TC. Therefore we get $\deg K = 2_{gtop} - 2, namely,$[5]

$$g = g_{top.}$$

Some general results

Let us fix some notations and give some definitions.

The degree of a map. Let C be an algebraic curve, and ω a smooth 2 form on C, such that $\int_C \omega = 1;$ the de Rham cohomology class $[\omega]$ may be regarded as an element in $H^2(C, Z),$ and actually provides a basis of that space, allowing an identification $H^2(C,Z) \square Z. If \ f: C^{'} \to C$ is a nonconstant holomorphic map between two algebraic curves, then $f^{\#}[\omega]$ is a nonzero element in $H^2(C,^{'} \mathbf{Z}),$ and there is a well defined integer n such that

$$f^{\#}[\omega] = n[\omega^{'}].$$

Where $\omega^{'}$ is a smooth 2-form on $C^{'}$ such that $\int_{C^{'}} \omega^t = 1. If \ p \in C$ we have

$$\deg f^*(p) = \int c_1(f^*[p]) = \int$$

$$h^o(L) - h^1(L) = \deg L - g + 1.$$

So that the map f takes the value p exactly n times, including multiplicities in the sense of divisors; we may say that f covers Cn *times*.[6] The integer n is called the degree if f .

This need not be true if the algebraic C is singular. However the Riemann-Roch theorem is still true (provided we know that a line bundle on a singular curve is!) with g the arithmetic genus.

Since two holomorphic functions of one variable which agree on a non discrete set are identical, and since $C^{'}$ is compact, the number of pints in $f^{-1}(p)$ is always finite.

Branch point: Given again a nonconstant holomorphic map $f: C^{'} \to C,$ we may find a coordinate z around any $g \in C^{'}$ and a coordinate ω around $f(q)$ such that locally f is described as

(13.4)

$$\omega = z^{r}.$$

The number $r-1$ is called the ramification index of $f\ at\, q(or\ at\ p = f(q)), and\ p = f(q)$ is said to be a branch point if $r(p) > 1.$ The branch locus of f is the divisor in $C^{'}$

$$B^{'} = \sum_{q \in C^{'}} (r(q)-1).q$$

Or its image in C

$$B = \sum_{q \in C^{'}} (r(q)-1).f(q).$$

For any $p \in C$ we have

$$f^{*}(p) = \sum_{q \in f^{-1}(p)} r(q).q$$

$$\deg f^{*}(p) = \sum_{q \in f^{-1}(p)} r(q) = n.$$

From these formulae we may draw the following picture. If $p \in C^{'}$ does not lie in the branch locus, then exactly n distinct points of $C^{'}$ are mapped to $f(p).$ which means that $f: C^{'} - B^{'} \to C - B$ is a covering map. [7] It $p \in C^{'}$ is a branch point of ramification index r-1, at p exactly r sheets of the covering join together.

There is a relation between the canonical divisors of $C' and C$ and the branch locus. Let η be a meromorphic 1-form on C, which can locally be written as

$$\eta = \frac{g(\omega)}{h(\omega)} d\omega$$

From (13.4) we get

$$f^*\eta = \frac{g(z^r)}{h(z^r)} dz^r = rz^{r-1} \frac{g(z^r)}{h(z^r)} dz$$

So that

$$ord_p f^*\eta = ord_{f(p)}\eta + r - 1.$$

This implies the relation between divisors

$$(f^*\eta) = f^*(\eta) + \sum_{p \in C'} (r(p) - 1).p.$$

On the other hand the divisor (η) is just the canonical divisor of C, so that

$$K_{C'} = f^* K_C + B^1.$$

A (holomorphic) covering map $f : X \to Y, with\, X$ connected, is a map such that each $p \in Y$ has a connected neighbourhood U such that $f^{-1}(U) = \cup_0 U_0$ is the disjoint union of open subsets of X which are biholomorphic to $U\, via\, f$. From this formula we may draw an interesting result. By taking degree we get

$$\text{Deg } K_C{}' = n \deg K_C + \sum_{p \in C^1} (r(p) - 1);$$

By using the Riemann-Hurwitz formula we obtain

(13.5) $$g(C^1) = n(g(C) = 1) + 1 + \frac{1}{2} \sum_{p \in C''} (r(p) - 1).$$

EXERCISES 13.1. Prove that if $f: C^{'} \to C$ is nonconstant, then $f^{*}: H^{0}(C, K_{C}) \to H^{0}(C^{''}, K_{C^{'}})$ is injective. (Hint: a nonzero element $\omega \in H^{0}(C, K_{C})$ is a global holomorphic 1-form on C which is different from zero at all pints in an open dense subset of C. Write an explicit formula for $f^{*}\omega$......)

Both equation (8.5) and the previous Exercise imply

$$g(C^{'}) \geq g(C).$$

The genus formula for plane curves. An algebraic curve C is said to be plane if it can be imbedded into $\mathbf{P}_{2}$. Its image in $\mathbf{P}_{2}$ is the zero locus of a homogenous polynomial; the degree d of this polynomial is by definition the degree of C. As a divisor, C is linearly equivalent to dH (indeed, since Pic $(\mathbf{P}_{2}) \cong Z.$ any divisor D on $\mathbf{P}_{2}$ is linearly equivalent to mH for some m; if D is effective, m is the number of intersection points between D and a generic hyperplane in $\mathbf{P}_{2}$, and this is given by the degree of the polynomial cutting D).

We want to show that for smooth plane curves the following relation between genus and degree holds.

(13.6) $g(C) = \dfrac{1}{2}(d-1)(d-2).$

(For singular plane curves this formula must be modified.) We may prove this equation by using the adjunction formula: C is imbedded into $\mathbf{P}_{2}$ as a smooth analytic hyper surfcae, so that

$$K_{C} = t^{*}(K_{\mathbf{P}_{2}} + C),$$

Where $t: C \to \mathbf{P}_{2}.$ Recalling that $K_{\mathbf{P}_{2}} = -3H$ we then have $K_{C} = (d-3)t^{*}H.$ we are actually using here a piece of intersection theory. The fact is that any k-dimensional analytic sub variety V of an n-dimensional complex manifold X determines a homology class $[V]$ in the homology group $H_{2k}(X, Z).$ Assume that X is compact, and let W

be an $(n-k)$ - dimensional analytic sub variety of X; the homology cap product $H_{2K}(X,Z) \cap H_{2n-2k}(X,Z) \to Z$, which is dual to the cup product in cohomology, associates the integer number $[V] \cap [W]$ with the two sub varieties. One may pick up different representatives $V' and W' of [V] and [W]$ such that $V' and W'$ meet transversally, i.e. they meet at a finite number of points; then the number $[V] \cap [W]$ counts the intersection points

In our case the number of intersection points is given by the number of solutions to an algebraic system, given by the equation of $C in \mathrm{P}_2$ (which has degree d) and the linear equation of a hyperplane. For a generic choice of the hyperplane, the number of solutions is d.

To carry on the computation, we notice that, as a divisor on $C, i^* H = C \cap H$, so that

$$\deg i^* H = d$$

and

$$\deg K_C = d(d-3) = 2g-2$$

Whence the formula (8.6).

Example 13.2. Consider the affine curve in C^2 having equation.

$$y^2 = x^6 - 1.$$

By writing this equation in homogeneous coordinates one obtain a curve in P_2 which is a double covering of P_1 branched at 6 points. By the Riemann-Hurwitza formula we may compute the genus, obtaining g=2. Thus the formula (8.6), which would yield g=10,fails in this case. This happens because the curve is singular at the point at infinity.

The residue formula. A meromorphic 1-form on an algebraic curve C is a meromorphic section of the canomical bundle K. Given a point $p \in C$, and a local holomorphic coordinate $z(p) = 0$, a meromorphic 1-form φ is locally written around p in the form $\varphi = f\,dz$, where f is a meromorphic function. Let a be coefficient of the z^{-1} term i nteh Laurent expansion of f around p, and let B a small disc around p; by the Cauchy formula we have

$$\alpha = \int_{\delta B} \varphi$$

So that the number a does not depend on the representation of φ. We call it the residue of φ at p, and denote it by $\operatorname{Re} s_p(\varphi)$.

Given a meromorphic 1form $\operatorname{Re} s_p(\varphi)$. its polar divisor is $D = \sum_i Pi,$

where the $p_i^{'}$s are the points where the local representatives of φ have poles of order 1.

PROPOSITIO N 13.3. Let $D = \sum_i Pi$ be the polar divisor of a meromorphic 1-form φ. Then $\sum_i \operatorname{Re} s_{p_i}(\varphi) = 0.$

PROOF. Choose a small disc B_i around each point p_i. Then

$$\sum_i \operatorname{Re} s_{p_i}(\varphi) = \int_{\delta \cup_i B_i} \varphi = -\int_{C - \cup_i B_i} d\varphi = 0.$$

a contradiction.

Otherwise one can directly identify the sections of L with meromorphic functions having (only) a single pole at p, since such functions can be developed around p, in the form

$$f(x) = \frac{a}{z} + g(z),$$

Where g is a holomorphic function, $a \in C$ should be identified with the projection of f onto k_p. (Here z is a local complex coordinate such that $z(p) = 0$.)

Check Your Progress

Notes

1. Prove: Any compact Riemann surface can be realized as a sub manifold of P_N for some N.

2. Prove: If M is a smooth 1-dimensional 2 analytic sub manifold of projective space P_n (i.e. M is the imbedding of a compact Riemann surface into P_n),and L is a line bundle on M , there is a divisor $D \, on \, M$ such that $L=[D]$.

3. Prove: Riemann-Roch theorem.

13.5 LET US SUM UP

1. Any compact Riemann surface can be realized as a sub manifold of P_N for some N.

2. A (1,1) form ω on a complex manifold is said to be positive if it can be locally written in the form $\omega = i\omega_{ij} dz^i \wedge d_z^{-j}$

This map actually depends on the choice of a basis of $|D|$; however, different choices correspond to an action of the group $\mathrm{P}Gl(N+1,\mathrm{C})\mathrm{P}_N$ and therefore produce isomorphic subvarieties of P_N

with ω_{ij} a positive definite hermitian matrix.

3. Any analytic subvariety of P_n is algebraic.

4. For any line bundle L on C one has $h^o(L) - h^1(L) = \deg L - g + 1.$

5. Let $D=\sum_i Pi$ be the polar divisor of a monomorphic 1-form φ.

Then $\sum_i \operatorname{Re} s_{p_i}(\varphi)=0.$

13.6 KEY WORDS

1. Polar divisor of a meromorphic
2. Kodaina embedding
3. Riemann-Roch Theorem
4. Riemann-Hurwitz formula

13.7 QUESTIONS FOR REVIEW

1. Prove Riemann-Roch theorem
2. Explain about general results about algebraic curves.
3. Prove : Let $D=\sum_i Pi$ be the polar divisor of a meromorphic 1-form φ. Then $\sum_i \operatorname{Re} s_{p_i}(\varphi)=0.$

13.8 SUGGESTIVE READINGS AND REFERENCES

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13.9 ANSWERS TO CHECK YOUR PROGRESS

1. See section 13.3
2. See section 13.3
3. See section 13.4

UNIT-14 ALGEBRIAC CURVES II

STRUCTURE

14.0 Objective

14.1 Introduction

14.2 The Jacobian variety

14.3 Elliptic curves

14.4 Nodal curves

14.5 Transforms of a curve

14.6 Normalization of a nodal plane curve

14.7 Let's Sum up

14.8 Keyword

14.9 Questions for review

14.10 Suggested readings and reference

14.11 Answer to check your prrogress

14.0 OBJECTIVE

In this unit we will learn and understand about The Jacobin variety, Elliptic curves, Nodal curves, Transforms of a curve, Normalization of a nodal plane curve.

14.1 INTRODUCTION

In this chapter we further study the geometry of algebraic curves. Topics covered include the Jacobian variety of an algebraic curve, some theory of elliptic curves, and the desingularization of nodal plane singular curves (this will involve the introduction of the notion of blow up of a complex surface at a point).

Notes

14.2 THE JACOBIAN VARIETY

A fundamental tool for the study of an algebraic curve is its Jacobian variety which we proceed now to define. Let be an m-dimensional complex vector space, and think of it as an abellian group. A lattice is a subgroup of of the form (14.1)

Where is a basis of as a real vector space. The quotient space has a natural structure of complex manifold, and one of abelian group, and the two structures are compatible, i.e., is a compact abelian complex Lie group. We shall a complex torus. Notice that by varying the lattice. one gets another complex torus which may not be isomorphic to the previous one (the complex structure may be different), even though the two tori are obviously diffeomorphic as real manifolds.

Example 14.1. if is an algebraic curve of genus the group classifying the line bundles on with vanishing first Chern class, has structure of complex torus of dimension since it can be represented as is a lattiece in . This is the Jacobian variety of . In what follows we shall construct this variety in a more explicit way.

Consider now a smooth algebraic curve of genus We shall call abelian differentials the global sections of (i.e, the global holomorphic 1-forms). If in abelian differential, we have this means that singles out a cohomology class and that

Moreover, since locally we have (14.3)

If is a smooth loop in the number depends only on the homology class of and the cohomology class of and expresses the pairing <,> between the Poincare dual spaces .

Pick up a basis of the 2g-dimensional free -module where the are smooth loops in and a basis We associate with these data the whose entries are the numbers

This is called the period matrix. Its columns are linearly independent over if for all Then also Since is a basis for this implies that is, So the columns of the period matrix generate a lattice, The quotient complex torus, is the Jacobian variety of

Define now the intersection matrix by letting (this is the valued "cap" or "intersection " product in homology).Notice that is an element in since the cup product in cohomoloyg is Pincare' dual to the cap product in homoloyg, for any abelian differentials we have The relations (14.2), (14.3) can then be written in the form (here denotes transposition, hermitian conjugation). In this form they are called Riemann bilinear relations.

A way to check that the construction of the Jacobi variety does not depend on the choices we have made is to restate it invariantly. Integration over cycles defines a map.

This map is injective: for a given and all then is homologous to the constant loop. Then we have the representation show hat Serre and Poincare dualities establish an isomorphism

The Abel map. After fixing a point in and a basis in we define a map (14.4)

By letting

Actually the value of will depend on the choice of the path from however, if are two paths, the oriented sum will define a cycle in homology, the two values will differe by an element in the lattice, and is a well defined point in

From (14.4) we may get a group homomorphism (14.5)

By letting

All of this depends on the choice of the base point note however that if deg then the choice is immaterial.

PROPOOSITION 14.3. (Abel's theorem) Two divisors are linearly equivalent if and only if Corollary 14.4. The Abel map is injective.

PROOF. If by the previous proposition as divisors, but since this implies . Abel's theorem may be stated in a fancier language as follows. Let be the subset of formed by the divisors of degree d, and let be the set of line bundles of degree One has a surjective map whose kernel is isomorphic to Then filters through a morphism and one has a commutative diagram

1 Notice that as sets for all values of

Moreover, the morphism is injective then that is, is trivial).

We can actually say more about the morphism namely, that it is a bijection. It is enough to prove that is surjective for affixed value of (cf. Previous footnote).

Let be the d-fold Cartesian product of with itself. The symmetric group of order acts on ; we call the quotient the d-fold symmetric product of can be identified with the set of effective divisors of of degree The map defines a map

Any local coordinate on yields a local coordinate system And the elementary symmetric functions of the coordinates yield a local coordinate system for Therefore the latter is a d-dimensional complex manifold. Moreover, the holomorphic map is invariant, hence it descends to a map which coincides with So the latter is holomorphic.

Exercise14.5. Prove that (Hint: write explicitly a morphism in homogeneous coordinates.)

The surjectivity of follows from the following fact, usally called Jacobi inversion theorem.

PROPOSITION 14.6. The map is surjective.

PROOF. Let with all the distinct, and let be a local coordinate centred in then is a local coordinate system around . If is near we have (14.6)

Where is the component of Consider now the matrix (14.7?)

We may choose so that and then subtracting a suitable multiple of from we may arrange that We next choose so that and arrange that and so on. In this way the matrix (14.7) is upper triangular. With these choices of the abelian differentials and of The points the Jacobian matrix is upper triangular as well, and since its diagonal elements are nonzero at s that at the point corresponding to our choices the Jacobian determinant is nonzero. This means that the determinant is not everywhere zero, and by Lemma 8.4 one concludes.

PROPOSITION 14.7. The map is generically one-to-one.

PROOF. Let and choose a divisor By Abel's theorem the fibre is formed by all effective divisors linearly equivalent to hence it is a projective space. But since dim the fibre of is generically 0-dimensional,so that generically it is a point.

This means that establishes a biholomorphic correspondence between a dense subset of and a dense subset of such maps are called birational.

COROLLARY 14.8. Every divisor of degree on an algebraic curve of genus is linearly equivalent to an effective divisor.

PROOF. Let We may write By mapping by Abel's map and taking a counter image in we obtain an effective divisor linearly equivalent to . Then is effective and linearly equivalent to .Corollary 14.9. Every elliptic smooth algebraic curve (i.e. every smooth algebraic curve of genus 1) is of the form for some lattice Proof. We have and the map concides with By Abel's theorem, if and only if there is on a meromorphic function such that but on there are no meromorphic functions with a single pole, so that is also surjective by Lemma 13.4 (this is a particular case of Jacobi inversion thereorem), hence it is bijective.

Corollary14.10. The canonical bundle of any elliptic curve is trivial.

Proof. We represent an elliptic curve as a quotient The (trivial)tangent bundle to is invariant under the action of , therefore the tangent bundle to is trivial as well.

Another consequence is that if is an elliptic algebraic curve and one chooses a point , the curve has a structure of abelian group, with playing the role of the identity element.

Notes

Jacobian varieties are algebraic. According to our previous discussion, any 1-dimensional complex torus is algebraic. This is no longer true for higher dimensional tori. However, the Jacobian variety of an algebraic curve is always algebraic.

Let be a lattice in   any point in the lattice singles out univoquely a cell in the lattice, and two opposite sides of the cell determine after identification a closed smooth loop in the quotient tours   This provides an identification

Let now   be a skew-symmetric   -bilinear form on   Since   canonically (check this isomorphism as an exercise),   may be regarded as a smooth complex-valued differential 2-form on T.

Proposition 14.11. The 2-form which on the basis   is represented by the intersection matrix   is a positive (1,1) form.

PROOF. If   are the real basis vectors in   generating the lattice, they can be regarded as basis in   They also generate   real vector fields on (after identifying   with its tangent space at 0 the   yield tangent vectors to at the pint corresponding to 0; by transporting them in all points of by left transport one gets vector files, which we still denote by   Let   be the natural local complex coordinates in ;the period matrix may be described as.

14.3 ELLIPTIC CURVES

Consider the curve   given by an equation (14.8)

So we are not only proving that the Jacobian variety of an algebraic curve is algebraic ,but, more generally, that any complex torus satisfying the Riemann bilinear relations is algebraic.

We are using the fact that if a smooth complex vector bundle   on a complex manifold   has a connection whose curvature has no (0.2) part, then the complex structure of   can be “lifted” to Cf..

Otherwise, we may use the fact that the image of the map   (the Neron-Severi group of   of subsection 6.5.1) may be represented as   i.e., as the group of integral 2-classes that are of Hodge type (1,1). The class of v is clearly of this type. Where are the standard coordinates in is a polynomial of degree 4. By writing the equation (14.8) in homogeneous coordinates, may be completed to an algebraic curve imbedded in a cubic curve in Let us assume that is smooth. By the genus formula we see that is an elliptic curve.

EXERCISE 14.1. Show that is a how here vanishing abelian differential on .After proving that all elliptic curves may be written in the form (14.8), this provides another proof of the triviality of the canonical bundle of an elliptic curve. (Hint: around each branch point, is a good local coordinate…)

The equation (14.8) moreover exhibits as a cover of which is branched of order 2 at the points where and at the point at infinity. One also checks that the point at infinity is a smooth point. We want to show that every smooth elliptic curve can be realized in this way.

So let be a smooth elliptic curve. If we fix a point and consider the exact sequence of sheaves on Proceeding as usual (Serre duality and vanishing theorem) one shows that is nonzero. A nontrivial section can be regarded as a global meromorphic function holomorphic in having a double pole at Moreover we fix nowhere vanishing holomorphic 1-form (which exists because K is trival). We have

We realize as these single s out a complex coordinate on the open subset of corresponding to the fundamental cell of the lattice . Then we may choose may be chosen in such a way that On the other hand, the meromorphic function is holomorphic outside and has a triple pole at We may choose constants such that

The line bundle is very ample, i.e., its complete linear system realizes the Kodaira imbedding of into projective space. By Riemann-Roch and the vanishing theorem we have so that is imbedded into To realize explicitly the imbedding we may choose three global sections corresponding to the meromorphic functions We shall see that these are related by a polynomial identity, which then expresses the equation cutting out We indeed have, for suitable constants So, that setting

So the meromorphic function in the left-hand side is holomorphic away from p, and has at p a simple pole. Such a function must be constant, otherwise it would provide an isomorphism of with the Riemann sphere.

Thus may be described as a locus in whose equation in affine coordinates is (14.9) For a suitable constant By a linear transformation on we may set and then by a linear transformation of we may set the two roots of the polynomial in the right- hand side of (14.9) to 0 and 1. So we express the elliptic curve in the standard form (Weiersta representation) 5 (14.10)

EXERCISE 14.2 Determine for what values of the parameter the curve (14.10) is smooth.

We want to claborate on this construction. Having fixed the complex coordinate, the function is basically fixed as well. We call it the Weierstra function. Its derivative is Notice that cannot contain terms of odd degree in its Laurent expansion, otherwise would be a nonconstant holomorphic function on so.

For suitable constants From this we see that satisfies the condition One usually writes for for the constant in the right-hand side.

In terms of this representation we may introduce a map is the set of isomorphism classes of smooth elliptic curves . Even though the Welerstatrft representation only provides the equation of the affine part of an elliptic curve, the latter is nevertheless completely characterized. It is indeed true that any affine plane curve can be uniquely extended to a compact cure by adding points at infinity, as one can check by elementary considerations. One shows that this map is bijective; in particular gets a structure of complex manifold. The number is called the j-invariant of the curve . We may therefore say that the moduli space is isomorphic to EXERCISE 14.3. Write the j-invariant as a function of the parameter in equation (14.10). Do you think that is a good coordinate on the moduli space ?

The holomorphic map

Imbeds into as the cubic curve cut out by the polynomial (we use the same affine coordinates as in the prevous representation). Since we have And the inverse of is the Abel map, - Having chosen at the point at infinity, In terms of this construction we may give an elementary geometric visualization of the group law in an elliptic curve. Let us choose as the identity element in . We shall denote by the element regarded as a group element By Abel's theorem, Proposition 14.3, we have that if and only if (indeed one may think that and one has Let be the equation of the equation of the line in through the points be the further intersectrion of this line with The function vanishes (of order one) only at the points and has a pole at .This pole must be of order three, so that the divisor of is The fancy coefficient 1728 comes from arithmetic geometry, where the theory is tailoted to work also for fields of characteristic 2 and 3.

By uniformization theory one can also realize this this moduli space as a quotient , where is the upper half complex plane. This is not contradictory in that the quotient is biholomorphic to (Notice that on the contrary, are not biholomorphic). One should bear in mind that we have identified with a quotient If then ,so that are collinear. Vice versa,if are collinear, is the divisor of the meromorphic function M, so that We have therefore shown that if and only if are collinear points in . Example 14.4. Let be an elliptic curve having a Weierstaraff represnetation is a double cover of ,branched at the three points. and at the point at infinity . The points are collinear, so that The two points The line through intersects, at the point at infinity, as one may check in homogeneous coordinates. So in this case the elements are one the inverse of the other, and , and is the further intersection of with the line going through if then So the branch points are 2-torsion elements in the group, .

14.4 NODAL CURVES

Notes

In this section we show how (plane) curve singularities may be resolved by a procedure called blowup. Blowup. Blowing up a point in a means replacing the point with all possible directions along which one can approach it while moving in the variety. We shall at first consider the blowup of at the origin; since this space is 2-dimentional, the set of all possible directions is a copy of Let be the standard coordinates in , and homogeneous coordinates in . The blowup of at the origin is the subvariety defined by the equation To show that is a complex manifold we cover with two coordinate charts, are the standard affine charts in , with coordinates and . is a smooth hypersurface in ,hence it is a complex surface. On the other hand if we put homogeneous coordinates then can be regarded as a open subset of the quadric in having equation so that is actually algebraic. our treatment of the blowup of an algebraic variety is basically taken from .

NODAL CURVES

Since is a subset of there are two projections (14.11) Which are holomorphic. If is a point (which means that hter is a unique line through so that Is a biholomorphism. On the contrary is the set of lines through the origin in The fibre of over a point is the line so that makes into the total space of a line bundle over . This bundle trivializes over the cover , and the transition function is so that the line bundle is actually the tautological bundle This construction is local in nature and therefore can be applied to any complex surface (two-dimensional complex manifold) at any point Let be a chart around with complex coordinates By repeating the same construction we get complex manifold with projections. And Is a biholomorphism, so that one can replace inside and get complex manifold with a projection which is abiholomorphism outside The manifold is the blowup of at The inverse image is a divisor in , called the exceptional divisor, and is isomorphic to The construction of the blowup show that is algebraic if is. EXAMPLE 14.1. The blowup of at a point is an algebraic surface (an example of a Del Pezzo surface): the manifold ,obtained by blowing up at the origin, is biholomorphic to minus a projective line

14.5 TRANSFORMS OF A CURVE

Let be a curve in containing the origin. We denote as before the blowup of at the origin and make reference to the diagram (14.11). notice that the inverse image contains the exceptional divisor is isomorphic to

so, according to a terminology we have introduce in a previous chapter, the map is a bi rational morphism.

DEFINITION 14.2 The curve is the total transform of . The curve obtained by taking the topological closure of is the strict transform of

We want to check what points are added to when taking the topological closure. To this end we must understand what are the sequence in which converge to 0 that are lifted by to convergent sequences. be a sequence of points in converging to 0; then is the point with Assume that for big enough one has (otherwise we would assume and would make a similar argument). Then converges if and only if has a limit, say h; in that case converges to the point This means that the lines joining 0 to approach the limit line having equation So a sequence convergent to 0 lifts to a convergent sequence in T if and only if the lines admit a limit lines in that case, the lifted sequence converges to the point of representing the line.

The strict transform meets the exceptional divisor in as many points as are the directions along which one can approach 0 on . Namely, as are the tangents at at 0. So, if is smooth at 0, its strict transform meets at one point. Every intersection point must be counted with its multiplicity; if at the point 0 the curve has coinciding tangents, then the strict transform meets the exceptional divisor at a point of multiplicity .

DEFINITION 14.3. Let the (affine plane) curve be given by equation We say that has multiplicity at 0 if the Taylor expansion of at 0 starts at degree .

Notes

This means that the curve has tangents at the point 0 (but some of them might coincide).By choosing suitable coordinates one can apply this notion to any point of a plane curve.

EXAMPLE 14.4. A curve is smooth at 0 if and only if its multiplicity at 0 is 1. The curves have multiplicity 2 at 0. The first two have two distinct tangents at 0, the third has a double tangent.

If the curve has multiplicity at 0 than it has tangents at 0, and its strict transform meets the exceptional divisor of points (notice however that those points are all distinct only if the tangents are distincts).

DEFINITION 14.5. A singular point of a plane curve is said to be nodal if at that point has multiplicity 2, and the two tangents to the curve at that point are distinct.

EXERCISE 14.6. With reference to equation (14,10), determine for what values of the curve has nodal singularity.

EXERCISE 14.7. Show that around a nodal singularity a curve is isomorphic to an open neighbourhood of the origin of the curve

EXAMPLE 14.8. (Blowing up a nodal singularity). We consider the curve having equation This curve has multiplicity 2 at the origin, and its two tangents at the origin have equations has a nodal singularity at the origin. We recall that is described as the locus

The projection is described as (14.12)

In respectively. By substituting the first of the representations (14.12) into the equation of we obtain the equation of the restriction of the total transform to Where is the equation of the exceptional divisor, so that the equation of the strict transform is By letting u=0 we obtain the points (0,0,1,1) and (0,0,1,-1) as intersection points of the strict transform with the exceptional divisor. By substituting the second representation is eq.(14.12) we obtain the equation of the total transform in the strict transform now has equation yielding the same intersection points. The total transform is a reducible curve, with two irreducible components which meet at two points.

EXERCISE 14.9. Repeat the previous calculation for the nodal curve
In particular show that the total transform is a reducible curve, consisting of the exceptional divisor and two more genus zero components, each of which meets the exceptional divisor at a point.

EXERCISE 14.10. (The cusp) Let be curve with equation This curve has multiplicity 2 at the origin where it has a double tangent. 10 Proceeding as in the previous example we get the equation so that does not meet in this chart. In the other chart the equation of is so that meets at the point (0,0,0,1); we have one intersection point because the two tangents to at the origin coincide.

The strict transform is an irreducible curve, and the total transform is a reducible curve with two components meeting at a (double) point.

Indeed this curve can be regarded as the limit for of the family of nodal curves which at the origin are tangent to the two lines

14. 6 NORMALIZATION OF A NODAL PLANE CURVE

It is clear from the previous examples that the strict transform of a plane nodal curve (i.e., a plane curve with only nodal singularities) is again a nodal curve, with one less singular point. Therefore after a finite number of blow-ups we obtain a smooth curve together with birational morphism is called the normalization of .

EXAMPLE 14.11. Let us consider the smooth curve having equation Projection onto the makes , into a double cover of branched at the points The curve can be completed to a projective curve simply by writing its equation in homogeneous coordinates and considering it as a curve we are thus compactifying by adding a point at infinity, which in this case is not a branch point. The equation of is This curve has genus 1 and is singular at infinity (as one could have already guessed since the genus formula for smooth plane curves does not work); indeed, after introducing affine coordinates (in this coordinates the point at infinity on the we have the equation Showing that is indeed singular at infinity. One can redefine the coordinates so that has equation . Showing that is a nodal curve. Then it can be desingularized as in Example 14.8.

A genus formula. We give here, without proof, a formula which can be used to compute the genus of the normalization of a nodal curve. Assume that has irreducible components and those singular points. Then: For instance, by applying this formula to Example 14.8, we obtain that the normalization is a projective line.

Check Your Progress

1. Prove: (Abel's theorem) Two divisors are linearly equivalent if and only if

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2. Prove: The map is surjective.

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4. Explain about elliptic curves.

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14.7 LET US SUM UP

1. Let be an m-dimensional complex vector space, and think of it as an abellian group. A lattice is a subgroup of of the form

Where is a basis of as a real vector space.

2. (Abel's theorem) Two divisors are linearly equivalent if and only if

3. The map is surjective.

4. Every divisor of degree on an algebraic curve of genus is linearly equivalent to an effective divisor.

5. The canonical bundle of any elliptic curve is trivial.

6. The curve is the total transform of . The curve obtained by taking the topological closure of is the strict transform of

14.8 KEY WORDS

The Jacobian variety

Abellian group

Cohomology

Abel's theorem

Elliptic curve

Nodel curve

14.9 QUESTIONS FOR REVIEW

1. Explain about Elliptic curves

2. Explain about nodal curves

3. Explain about transforms of a curve

14.10 SUGGESTIVE READINGS AND REFERENCES

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14.11 ANSWERS TO CHECK YOUR PROGRESS

1. See section 14.3

2. See section 14.3

3. See section 14.4